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**A LIE-ALGEBRAIC APPROACH TO THE  
LOCAL INDEX THEOREM ON  
COMPACT HOMOGENEOUS SPACES**

A Dissertation in  
Mathematics  
by  
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# ABSTRACT

Using a K-theory point of view, R. Bott related the Atiyah-Singer index theorem for elliptic operators on compact homogeneous spaces to the Weyl character formula. This dissertation explains how to prove the local index theorem for compact homogenous spaces using Lie algebra methods. The method follows in outline the proof of the local index theorem due to N. Berline and M. Vergne. But the use of B. Kostant's cubic Dirac operator in place of the Riemannian Dirac operator leads to substantial simplifications. An important role is also played by the quantum Weil algebra of A. Alekseev and E. Meinrenken.

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# LIST OF SYMBOLS

$\mathcal{A}$ , 134	$\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ , 89	$\exp_*$ , 54
$A_{\text{bas}}$ , 88	$\mathcal{D}_{\pm}$ , 120	$\exp_{\text{Cl}}$ , 75
$A_{\mathfrak{k}\text{-bas}}$ , 89	$\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$ , 107	$\exp_V$ , 60
$A^{\mathfrak{g}}$ , 88	$D(A)$ , 52	$f \sim g$ , 18
$A^{\mathfrak{k}}$ , 88	$D(G)$ , 18	$f _T$ , 35
$A_{\text{hor}}$ , 88	$D(G)^G$ , 59	$\text{Fr}(F)$ , 94
$A_{\mathfrak{k}\text{-hor}}$ , 88	$D(\mathfrak{g})$ , 59	$\text{Fr}_O(F)$ , 94
$\hat{A}$ , 124	$D(\mathfrak{g})^G$ , 59	$\text{Fr}_{SO}(F)$ , 94
$\text{ad}$ , 12	$D_e$ , 53	
$\text{Ad}$ , 12	$\Delta_G$ , 14	$\bar{\mathfrak{g}}$ , 78
$\widetilde{\text{Ad}}$ , 114	$\Delta_{\mathfrak{g}}$ , 59	$\mathfrak{g}_{\mathbb{C}}$ , 30
$\text{ad}_s$ , 72	$\hat{\Delta}_{\mathfrak{g}}$ , 85	$\mathfrak{g}_{\mathbb{C}, \alpha}$ , 30
$\text{Alt}_{\lambda}$ , 36	$\delta_G$ , 54	$\hat{\mathfrak{g}}$ , 79
$\text{Alt}(\mu)$ , 143	$\delta_{\mathfrak{g}}$ , 54	$\widehat{G}$ , 20
	$\delta_{ij}$ , 24	$\gamma$ , 76
$[ , ]_{\mathfrak{g}}$ , 78	$D(\mathfrak{g}, \mathfrak{k})$ , 91	$\gamma^{\mathfrak{g}}$ , 89
$[ , ]_s$ , 70, 72	$\text{diag}_W$ , 91	$\gamma^{\mathfrak{p}}$ , 89
	$\mathcal{D}'(M)$ , 56	$\Gamma(S)$ , 4
$c$ , 101	$\text{Duf}$ , 56	$\Gamma^2 E(G)$ , 38
$\mathbb{C}_{\mu}$ , 142	$\overline{\text{Duf}}$ , 59	$\text{gr } A$ , 57
$C(G, \mathbb{C})_{\text{fin}}$ , 25		
$C^{\infty}(G, \mathbb{C})_{\text{fin}}$ , 25	$E$ , 125	$\text{HP}$ , 95
$\text{ch}$ , 124	$e$ , 11	$H_p P$ , 95
$\chi_u$ , 19	$e_I$ , 69	$h_t$ , 16
$\text{Cl}(M)$ , 100	$e_I^{\varepsilon}$ , 70	
$\text{Cl}(M)$ , 100	$\mathcal{E}'_e(A)$ , 53	$\text{ind}$ , 120
$\text{Cl}(V)$ , 68	$\mathcal{E}'_0(\mathfrak{g})$ , 54	$\text{Ind}_K$ , 141
$\text{Cl}(\mathfrak{n})$ , 68	$\mathcal{E}'_0(\mathfrak{g})^G$ , 56	$\text{ind}_s$ , 120
$\text{Cl}^k(V)$ , 70	$\mathcal{E}'_e(G)$ , 54	$\iota_w$ , 73
$\text{Cl}_k(V)$ , 69	$\mathcal{E}'_e(G)^G$ , 56	$\iota_X$ , 78, 84
$\text{Cl}(V)$ , 68	$\varepsilon$ , 53	$j(X)$ , 56, 131
	$\varepsilon_G$ , 59	$j_{\mathfrak{k}}(X)$ , 131
$\mathcal{D}$ , 79, 81	$\varepsilon_{\mathfrak{g}}$ , 59	$j_{\mathfrak{g}/\mathfrak{k}}(X)$ , 131

$\kappa$ , 13	$P \times_p iE$ , 94	$T_K^*M$ , 139
$L_{cl}^2(G, \mathbb{C})$ , 26	$\mathcal{Q}$ , 85	$\mathrm{tr}_V$ , 135
$\lambda$ , 76	$q$ , 70	$\mathrm{tr}_{\mathfrak{g}}$ , 60
$\lambda^p$ , 91, 126	$R_f$ , 141	$\check{u}$ , 19
$\Lambda_{\mathrm{coroot}}$ , 32	$R_f^\pm$ , 141	$\mathcal{U}(\mathfrak{g})$ , 21
$\Lambda_e$ , 29	$R(G)$ , 37	$\mathcal{U}_k(\mathfrak{g})$ , 57
$\Lambda_{\mathfrak{g}}$ , 33	$\widehat{R}(G)$ , 38	$[V]$ , 36
$\Lambda_T$ , 29	$\rho$ , 32	$V(\mu)$ , 36
$L_X$ , 78, 84	$\rho_{\mathfrak{g}}$ , 91	$\mathrm{vol}$ , 14
$\nu$ , 126	$\mathbb{S}$ , 103, 114	$VP$ , 94
$\Omega$ , 23	$\mathbb{S}^\pm$ , 76	$V_p P$ , 94
$\widehat{\Omega}_{\mathfrak{g}}$ , 82	$\mathbb{S}_n$ , 68	$W$ , 28
$\Omega_{\mathrm{bas}}(P; E)$ , 96	$S(\mathfrak{g})$ , 54	$W(\mathfrak{g})$ , 83, 98
$\Omega(M; P \times_{\vee} E)$ , 96	$S^k(\mathfrak{g})$ , 57	$W(\mathfrak{g}, \mathfrak{k})$ , 91
$O(t^n)$ , 17	$\sigma$ , 58	$\mathcal{W}(\mathfrak{g})$ , 78
$O(t^\infty)$ , 18	$\sigma_k$ , 57	$\mathcal{W}(\mathfrak{g}, \mathfrak{k})$ , 89
$P_0$ , 122	$\mathrm{Span}_{\mathbb{R}}$ , 58	$\overline{X}$ , 78
PBW, 54	Str, 122	$\widehat{X}$ , 79
$\Phi$ , 32	supp, 142	$\widetilde{X}$ , 11, 94
$\Phi^+$ , 32	$\tau$ , 21, 52	$\widetilde{X}_e$ , 12
$\pi$ , 126	$T(\mathfrak{g})$ , 21	$\Xi$ , 128
$P_{\mathrm{Spin}}(M)$ , 102	$\Theta_V$ , 134	$\mathcal{Z}(\mathfrak{g})$ , 23
$P \times_G E$ , 93	$\Theta_T$ , 134	

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*Spectral point of view is the one which appears  
from experiments, when you study the universe,  
this is no fantasy.*

— A. Connes [49]

# 1

## INTRODUCTION

FROM the sound waves of a drum to a quantum particle trapped in a box, many fundamental physical systems can be studied by solving the eigenvalue problem of the Laplacian under Dirichlet boundary conditions:

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{on } U, \\ \phi = 0, & \text{on } \partial U, \end{cases} \quad (1.0.1)$$

where  $U$  is a bounded open subset of a Euclidean space. It is a standard result of elliptic partial differential operator theory that there is an orthonormal basis for  $L^2(U)$  consisting of solutions to Equation 1.0.1. Moreover, the basis vectors can be ordered in such a way that their corresponding eigenvalues  $\{\lambda_k\}_{k=1}^\infty$  form an unbounded sequence of nondecreasing positive real numbers.

A significant question is “how fast does  $\lambda_k$  grow?” This simple question is related to one of the most fascinating chapters in the history of science — the problem of blackbody radiation and the birth of quantum physics. Consider a cube of side length  $L$  with hollow interior, and walls that completely block any light. The intensity of the electromagnetic waves inside the cube that is under thermodynamic equilibrium was studied in the late 19th century, and it led to the discovery of Planck’s constant  $h$ . To calculate the intensity, one needs to calculate the total number of standing waves that have frequency less than  $f$ ; call that number  $N(f)$ . The standing waves are the solutions of Equation 1.0.1 where  $U$  is the interior of the cube. Using the method of separation of variables, one finds that the eigenvalues are of the form

$$\lambda = \frac{\pi^2}{L^2}(\ell^2 + m^2 + n^2),$$

where  $\ell, m, n$  are positive integers. Physically,  $\lambda$  is related to the

frequency  $f$  of the standing wave by

$$\lambda = (2\pi f)^2.$$

Thus,  $N(f)$  is equal to the number of lattice points in the positive octant of  $\mathbb{R}^3$  that are bound by radius  $R := 2fL$ . Thus,  $N(f)$  can be approximated by  $1/8$  of the volume of the ball that has radius  $R$ . The error of this estimate vanishes as  $f$  tends to infinity, and we have

$$N(f) = \frac{4\pi}{3} f^3 L^3 + o(f^3). \quad (1.0.2)$$

Because of the thermal origin of the blackbody radiation, this relation is expected<sup>1</sup> to hold even when  $U$  is an arbitrarily shaped cavity. In other words, we expect the following to hold for any bounded open subset  $U$  of  $\mathbb{R}^3$ :

$$\lim_{f \rightarrow \infty} \frac{N(f)}{f^3} = \frac{4\pi}{3} \text{vol}(U), \quad (1.0.3)$$

where  $\text{vol}(U)$  is the volume of  $U$ .

Indeed, H. Weyl [94, 95] proved the asymptotic law

$$\lim_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\text{vol}(U)}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)}, \quad (1.0.4)$$

where  $U$  is a bounded open subset of  $\mathbb{R}^n$  with  $n = 2$  or  $3$ . Equation 1.0.4 is known as *Weyl's law*. L. Gårding [46] proved it for higher-dimensions (for generic elliptic operators). For the Laplacian on a closed Riemannian manifold  $U$ , the same law was proved by S. Minakshisundaram and Å. Pleijel [75], even for generic elliptic operators by J. J. Duistermaat and V. W. Guillemin [33].

Weyl's law tells us, in particular, that if we know the spectrum of the Laplacian then from it we can calculate the volume of the ambient space  $U$ . In this regard (and for other more) Weyl's law, as N. Higson puts it [56, p. 456], is the first theorem of noncommutative geometry.

**HEAT TRACE.** Weyl's law can be reformulated as an asymptotic behavior of the *heat trace*

$$Z(t) = \text{tr}(e^{t\Delta}) = \sum_{k=1}^{\infty} e^{-t\lambda_k}. \quad (1.0.5)$$

It resembles the “partition function” in physics — a function that is often invariant under the symmetry of the physical system it describes. (Let us not worry about the convergence of the sum at this point.) The relation between  $Z(t)$  and the number of eigenvalues under certain cutoff becomes evident if we write  $Z(t)$  as

$$Z(t) = \lim_{N \rightarrow \infty} \sum_{k=1}^N e^{-t\lambda_k}.$$

<sup>1</sup> It was H. A. Lorentz [71] who formally posed this as a mathematical problem.

In the limit  $t \rightarrow 0+$ , the partial sum

$$\sum_{k=1}^N e^{-t\lambda_k}$$

converges to  $N$ , which is the number of the eigenvalues from  $\lambda_1$  to  $\lambda_N$ . Clearly the infinite sum  $Z(t)$  tends to infinity as  $t \rightarrow 0+$ ; but with an appropriate power of  $t$  multiplied to  $Z(t)$ , we have:

$$\lim_{t \rightarrow 0+} t^{n/2} Z(t) = \frac{\text{vol}(U)}{(4\pi)^{n/2}}, \quad (1.0.6)$$

which is equivalent to Weyl's law (see [85, Thm. 8.16, p. 115]).

When S. Minakshisundaram and Å. Pleijel proved Weyl's law for closed Riemannian manifolds, they showed that the heat trace has an asymptotic<sup>2</sup> expansion

$$Z(t) \sim \frac{1}{(4\pi t)^{n/2}} (a_0 + a_1 t + a_2 t^2 + \dots) \quad (1.0.7)$$

for  $t \rightarrow 0+$ , and that

$$a_0 = \text{vol}(M),$$

where  $M$  is the underlying manifold. Later, H. P. McKean Jr. and I. M. Singer [74] calculated (in addition to  $a_0$ )  $a_1$  and  $a_2$ ; according to their result,

$$a_1 = \frac{1}{6} \int_M S, \quad (1.0.8)$$

where  $S$  is the scalar curvature of  $M$ ; the coefficient  $a_2$  is not as simple as the earlier ones, but it is the integral of a polynomial of degree 2 in the components of the Riemann curvature tensor with respect to an orthonormal framing (the polynomial is invariant under the choice of the framing). The higher order coefficients are, in general, extremely hard to calculate.

What is interesting is that, if  $M$  is 2-dimensional, then, applying the Gauss-Bonnet<sup>3</sup> theorem [15, 45] to the result 1.0.8 gives us

$$a_1 = \frac{\pi}{3} \chi(M),$$

where  $\chi(M)$  is the Euler characteristic of  $M$ . Thus, even though the asymptotic expansion 1.0.7 for  $Z(t)$  is, a priori, determined by the analytical properties of the Laplacian, the coefficient  $a_1$  is completely determined by the topology of  $M$  and is independent of the metric.

**LOCAL INDEX THEOREM.** The story becomes even more interesting as we consider the Dirac operator, which is the “square root” of the Laplacian. As P. A. M. Dirac [29] found out, the natural vector bundle for the Dirac operator is the spinor bundle (whose fibers are irre-

<sup>2</sup> The asymptotic expansion 1.0.7 means that, for each nonnegative integer  $N$ , we have  $(4\pi t)^{n/2} Z(t) - \sum_{k=0}^N a_k t^k = o(t^N)$  for  $t \rightarrow 0+$ .

<sup>3</sup> O. Bonnet [15] writes that the theorem had also been proved by J. Binet.

ducible modules over the Clifford algebra generated by the tangent space).

Suppose  $\mathcal{D}$  is a Dirac operator on the sections of the spinor bundle  $S$  over a compact even-dimensional Riemannian manifold  $M$ . By the representation theory of Clifford algebras, the spinor bundle, in this case, is naturally bi-graded and so is its space of sections:

$$\Gamma(S) = \Gamma(S)^+ \oplus \Gamma(S)^-.$$

In the most interesting cases, the Dirac operator is an odd operator:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

Moreover, its spectrum is an unbounded discrete subset of  $i\mathbb{R}$ , each eigenvalue occurring with finite multiplicity. The space  $\Gamma^2(S)$  of square-integrable sections of  $S$  admits a Hilbert space direct sum decomposition into the eigenspaces of  $\mathcal{D}$ . (We shall review all these in detail in Chapter 8.) Now consider the operator  $e^{t\mathcal{D}^2}$ ,  $t \in ]0, \infty[$ , and its *super-trace*, that is, the ordinary trace over the even domain minus the trace over the odd domain:

$$\text{Str}(e^{t\mathcal{D}^2}) = \text{tr}(e^{t\mathcal{D}_- \mathcal{D}_+}) - \text{tr}(e^{t\mathcal{D}_+ \mathcal{D}_-}).$$

It turns out that there is a “super-symmetry” among the nonzero eigenvalues of  $\mathcal{D}^2$  in that the multiplicity in the even domain is equal to that in the odd domain; as a result, the super-trace of  $e^{t\mathcal{D}^2}$  is equal to the difference between the dimension of the kernel of  $\mathcal{D}$  in the even domain and that in the odd domain; this quantity is what is called the *graded index* of  $\mathcal{D}$ :

$$\text{Ind}_s \mathcal{D} = \dim \ker(\mathcal{D}_+) - \dim \ker(\mathcal{D}_-).$$

In short,

$$\text{Ind}_s \mathcal{D} = \text{Str}(e^{t\mathcal{D}^2}).$$

This is called the *McKean-Singer formula*, for such a phenomenon was first observed by H. P. McKean Jr. and I. M. Singer in their above mentioned work regarding the asymptotic expansion of the heat trace. The operator  $e^{t\mathcal{D}^2}$  admits an integral kernel, that is, a function

$$(t, x, y) \mapsto k_t(x, y) \in \text{End}(S_y, S_x)$$

for  $t \in ]0, \infty[$  and  $(x, y) \in M \times M$  such that

$$(e^{t\mathcal{D}^2} \sigma)(x) = \int_M k_t(x, y) \sigma(y) \text{vol}_y$$

for  $\sigma$  in  $\Gamma(S)$ . Here  $\text{vol}_y$  is the Riemannian volume form of  $M$  at  $y$ . In terms of the integral kernel,

$$\text{Ind}_s \mathcal{D} = \text{Str}(e^{t\mathcal{D}^2}) = \int_M \text{Str}(k_t(x, x)) \text{vol}_x.$$

Because of this,  $\text{Str}(k_t(x, x)) \text{vol}_x$  is called the *index density* of  $\mathcal{D}$ . R. T. Seeley's work [87] shows that  $k_t(x, x)$  admits an asymptotic expansion of the form

$$k_t(x, x) \sim \frac{1}{(4\pi t)^{n/2}} (a_0(x) + a_1(x)t + a_2(x)t^2 + \dots)$$

for  $t \rightarrow 0+$ , where  $n = \dim(M)/2$ . Then,

$$\text{Str}(k_t(x, x)) \sim \frac{1}{(4\pi t)^{n/2}} (\text{Str}(a_0(x)) + \text{Str}(a_1(x))t + \text{Str}(a_2(x))t^2 + \dots).$$

M. Atiyah, R. Bott, and V. K. Patodi [8] demonstrated that the leading nonzero term comes from  $a_{n/2}$  and that (in our simple setting)

$$\text{Str}(a_{n/2}) \text{vol} = \hat{A}$$

where  $\hat{A}$  is the Hirzebruch  $\hat{A}$ -class on  $M$ . Thus, the index density satisfies the asymptotic equality

$$\text{Str}(k_t(x, x)) \text{vol} = \hat{A} + O(t), \quad (1.0.9)$$

and thus,

$$\text{Ind}_s \mathcal{D} = \int_M \hat{A} + O(t).$$

Since the left-hand side is independent of  $t$ , we have

$$\text{Ind}_s \mathcal{D} = \int_M \hat{A}, \quad (1.0.10)$$

which is the celebrated Atiyah-Singer index theorem [10] in its simplest setting. Once again, we have an equation whose one side is analytical in nature while the other side is topological. The Atiyah-Singer index theorem is a far-reaching statement that generalizes the Gauss-Bonnet-Chern theorem [23], the Riemann-Roch-Hirzebruch theorem [57, 82, 84], the Hirzebruch signature theorem [58], and the Lefschetz fixed point theorem [70]; see [7; 12, § 4.1; 85, Ch. 10]. (M. Atiyah's account on the history leading to the index theorem can be found in [5].)

What we have described here is the so-called *heat kernel proof* of the index theorem. This is not the original approach took by M. Atiyah and I. M. Singer. But the heat kernel proof is stronger in that we have a *local index theorem*, namely, Equation 1.0.9, which says that the leading nonzero term in the asymptotic expansion of the index density is completely determined by the topology. Because of its local nature, the local index theorem is primed for generalizations to noncompact manifolds.

THE THESIS. Soon after M. Atiyah and I. M. Singer proved the index theorem, R. Bott [16] examined the special case of homogeneous spaces  $G/K$  where  $G$  is a compact connected Lie group and  $K$  is a closed connected subgroup. Using H. Weyl's theory (integral formula, character formula, and so on) R. Bott was able to verify the

index theorem by-passing much of the analytic or topological arguments. Expectation of such simplification for the *local* index theorem is the motivation behind this dissertation.

It seems, however, that the known proofs for the local index theorem — such as [8, 13, 47] — in themselves do not simply even if we restrict the manifolds under consideration to compact homogeneous spaces. What is common in the aforementioned proofs is that they use the Riemannian Dirac operator (see Section 6.2.6 for definition). The main thesis of this dissertation is then that the use of B. Kostant’s *cubic Dirac operator* [65] in place of the Riemannian Dirac operator leads to substantial simplifications; A. Alekseev and E. Meinrenken’s *quantum Weil algebra* [2, 3] also plays an important role.

To give a quick sketch, take a bi-invariant metric on  $G$ ; let  $\mathfrak{g}, \mathfrak{k}$  be the Lie algebras of  $G$  and  $K$ , respectively, and let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ . A. Alekseev and E. Meinrenken introduced [2] the *quantization map*

$$\mathcal{Q}: (S(\mathfrak{g}) \otimes \wedge(\mathfrak{p}))^K \rightarrow (\mathcal{U}(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^K. \quad (1.0.11)$$

Here  $S(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{g})$ , respectively, are the symmetric algebra and the universal enveloping algebra generated by  $\mathfrak{g}$ ;  $\wedge(\mathfrak{p})$  and  $\text{Cl}(\mathfrak{p})$ , respectively, are the exterior algebra and the Clifford algebra generated by  $\mathfrak{p}$ ; the decoration  $(\cdot)^K$  denotes the subspace of  $K$ -invariants in the respective algebra. The quantization map  $\mathcal{Q}$  is a vector space isomorphism given by the graded symmetrization of certain generators. Its image  $(\mathcal{U}(\mathfrak{g}) \otimes \text{Cl}(\mathfrak{p}))^K$  is called the *relative Weil algebra* for the pair  $(\mathfrak{g}, \mathfrak{k})$ . The algebraic structure of the relative Weil algebra singles out an element  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ , which happens to be B. Kostant’s cubic Dirac operator. (The details will be reviewed in Chapter 5.)

What this dissertation demonstrates is then the following. Let  $E$  be a finite-dimensional  $\text{Cl}(\mathfrak{p})$ -module, and let  $\Gamma(G \times E)^K$  be the space of  $K$ -equivariant sections of the trivial bundle  $G \times E \rightarrow G$ . The elements of the relative Weil algebra can be naturally identified as  $G$ -equivariant differential operators on  $\Gamma(G \times E)^K$ . Now there is a vector space isomorphism between  $\Gamma(G \times E)^K$  and the space  $\Gamma(E(G))$  of sections of a certain<sup>4</sup>  $K$ -equivariant bundle  $E(G) \rightarrow G/K$ ; and there is a differential operator  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  on  $\Gamma(E(G))$  such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(G \times E)^K & \xrightarrow{\mathcal{D}(\mathfrak{g}, \mathfrak{k})} & \Gamma(G \times E)^K \\ \parallel \wr & & \parallel \wr \\ \Gamma(E(G)) & \xrightarrow{\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}} & \Gamma(E(G)) \end{array}$$

<sup>4</sup> Because the inner product on  $\mathfrak{g}$  is bi-invariant, the adjoint representation of  $\mathfrak{k}$  on  $\mathfrak{p}$  is antisymmetric. Hence, we have a Lie algebra homomorphism  $\mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{p}) \simeq \mathfrak{spin}(\mathfrak{p}) \subseteq \text{Cl}(\mathfrak{p})$ . This induces a  $K$ -action on  $E$ . The bundle  $E(G)$  is then the  $K$ -orbit space of  $G \times E$  under the right  $K$ -action  $(g, v) \cdot k = (gk, k^{-1} \cdot v)$ . Details of this construction will be reviewed in Sections 5.1 and 6.1.

This operator  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  is a Dirac operator on  $G/K$ , but it is *not* the Riemannian Dirac operator which was used, for instance, in E. Getzler's proof [47] of the local index theorem. Our strategy is to deduce the heat kernel of  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}^2$  via the heat kernel of  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$ . To that end, we consider the preimage  $L$  of  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$  under the quantization map 1.0.11. This preimage has, as an element of  $(S(\mathfrak{g}) \otimes \wedge(\mathfrak{p}))^K$ , a natural identification as a differential operator on  $\Gamma(\mathfrak{g} \times E)^K$ . We shall show that the algebraic relation

$$\mathcal{Q}(L) = \mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$$

brings about a geometric equation involving the Laplacian  $\Delta_{\mathfrak{g}}$  on the Euclidean space  $\mathfrak{g}$  and the exponential chart near the identity of  $G$  (Proposition 7.2.11); the equation is so simple that the asymptotic expansion of the heat kernel of  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$  follows immediately. What remains is to deduce from this the heat kernel of  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}^2$ . We follow, in outline, the approach of N. Berline and M. Vergne [13]; what differs (aside from the already mentioned use of the cubic Dirac operator) is that the natural principal  $K$ -bundle  $G \rightarrow G/K$  is used instead of a principal  $\text{Spin}(\mathfrak{p})$ -bundle over  $G/K$ ; the benefit is that, owing to the homogeneity of  $G$ , almost all calculations are brought down to the level of Lie algebras, and this further eases the proof.

OUTLINE. We begin by reviewing the theory of compact Lie groups, focusing on the those parts that are relevant to the Laplacian associated to a bi-invariant metric. This is done in Chapter 2.

Chapter 3 is meant to be a warm-up with the heat trace of the Laplacian. We verify the volume formula of Harish-Chandra using Weyl's law and the Euler-Maclaurin formula.

We then tackle, in Chapter 4, the asymptotic expansion of the heat kernel of the Laplacian using Lie algebra methods; more precisely, we use the Duflo isomorphism. The efficiency of the Lie-algebraic approach becomes manifest.

Chapter 5 gives an extra supply of algebra, namely, the theory of quantum Weil algebra, which is necessary for our Lie-algebraic proof of the local index theorem

Chapter 6 shows how the elements of the quantum Weil algebra are naturally identified as equivariant differential operators on Clifford module bundles.

Having made the necessary preparations, we derive, in Chapter 7, the asymptotic expansion of the heat kernel of the square of the Kostant-Dirac operator using the quantization map between the classical and the quantum Weil algebra. The calculation is a modest update of what is done in Chapter 4.

Chapter 8 is the high point of this dissertation where we present a Lie-algebraic proof of the local index theorem for compact homogeneous spaces.

In the final chapter, we take a slightly different view point on



the index of the Dirac operator, namely, as a distribution associated to a transversally elliptic operator. We prove a pleasant theorem (Theorem 9.2.17) that involves the Duflo isomorphism and Chern-Weil theory.

OPEN QUESTIONS. Below are some questions that arise from this work, beginning with the immediate ones:

- (1) *Can the Lie algebra method be generalized to symmetric spaces of noncompact type?* Suppose that  $G/K$  is a symmetric space of noncompact type (for instance, a hyperbolic space). The index theory for the Dirac operator on  $G/K$  has featured in representation theory since the pioneering work of R. Parthasarathy [77]. M. Atiyah and W. Schmid [9] used Atiyah's  $L^2$ -form of the index theorem to give a geometric account of the construction of discrete series representations. In more recent work, V. Lafforgue [67] used the  $L^2$ -index theorem to show that the Baum-Connes conjecture for  $G$  also leads to a geometric description of the discrete series using Dirac operators and harmonic spinors. In both cases, general techniques of analysis and Riemannian geometry are used to establish the index theorem; but it is natural to guess that the Lie-algebraic approach demonstrated in this dissertation should extend to symmetric spaces. A simplified proof like the one sketched above would considerably simplify the connection between  $K$ -theory, Dirac operators, and the discrete series. Some difficulties are immediately apparent, such as the fact that the natural Laplacian on  $G$  (the Casimir operator) is not elliptic, but there are good indications that they can be overcome.
- (2) *Does Theorem 9.2.17 hold for general principal bundles?* Let  $K$  be a compact Lie group, and let  $P$  be a principal  $K$ -bundle over a compact manifold  $M$ . A Dirac operator on  $M$  can be lifted — using a connection on  $P$  — to a transversally elliptic operator  $\mathcal{D}$ . The definition of the distributional index of  $\mathcal{D}$  applies to this case. An obvious generalization of Theorem 9.2.17 is then to replace  $G/K$  with  $M$  in Equation 9.2.18.
- (3) *Can the Kostant-Dirac operator be used in the proof of the local index theorem in its full generality?* Let  $M$  be a compact spin manifold of dimension  $n$ . Let  $\text{Fr}_{\text{SO}}(M)$  be the principal  $\text{SO}(n)$ -bundle of oriented orthonormal frames for the tangent bundle of  $M$ . Let  $P_{\text{Spin}}(M) \rightarrow \text{Fr}_{\text{SO}}(M)$  be a spin structure for  $M$ . (Hence,  $P_{\text{Spin}}(M)$  is a principal  $\text{Spin}(n)$ -bundle that is a fiber-wise double covering of  $\text{Fr}_{\text{SO}}(M)$ .) The principal bundle  $P_{\text{Spin}}(M)$  plays a major role in N. Berline and M. Vergne's proof of the local index theorem. Is there a room for the Kostant-Dirac operator  $\mathcal{D}_K$  associated to the compact Lie group

$K = \mathrm{SO}(n)$ ? (For the definition of  $\mathcal{D}_K$ , see Section 5.2.1.) For instance, suppose  $\mathcal{D}_M$  is a Dirac operator on  $M$ . This can be lifted to a transversally elliptic operator on  $P_{\mathrm{Spin}}(M)$ . A suggestion by N. Higson is to augment  $\mathcal{D}_M$  with  $\mathcal{D}_K$  so that we would have an elliptic operator; would this streamline N. Berline and M. Vergne's proof?

- (4) *What is the relationship between the Getzler symbol map and the quantization map  $\mathcal{Q}$ ?* Let  $M$  be a compact spin manifold, and let  $\mathcal{D}$  be the Riemannian Dirac operator on some Clifford module bundle  $S \rightarrow M$ . E. Getzler, in his proof [47] of the local index theorem, constructed a symbol map from the filtered algebra  $\Psi$  of pseudo-differential operators on  $\Gamma(S)$  to the associated graded algebra of  $\Psi$ . When  $M$  is a compact homogeneous space  $G/K$ , the Getzler symbol map gives a vector space isomorphism

$$\sigma : (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K \rightarrow (S(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K.$$

This map is not the inverse of the quantization map 1.0.11. But is there a simple relation between the two maps? If there is, it could lead to a generalization of  $\mathcal{Q}$  for differential operators on Clifford module bundles over generic compact spin manifolds, with which one can hope to simplify the proof of the local index theorem in its full generality.

- (5) *Does the index theorem imply the Duflo isomorphism?* The restriction of the quantization map 1.0.11 yields the Duflo isomorphism  $(S\mathfrak{g})^{\mathfrak{g}} \simeq \mathbb{Z}(\mathfrak{g})$ . (This was shown by A. Alekseev and E. Meinrenken [3].) This underlies in our proof of the local index theorem on compact homogeneous spaces. One could ask whether the converse would hold; can we deduce the Duflo isomorphism from the index theorem?
- (6) *Can the Euler-Maclaurin formula explain the appearance of the  $\hat{A}$ -class in the index formula?* Both the Euler-Maclaurin formula 3.1.1 and the Hirzebruch  $\hat{A}$ -class involves the power series of

$$\frac{x}{1 - e^{-x}}.$$

Is this a coincidence, or is there a deeper connection between them?

BACKGROUND MATERIALS. The mathematics we take for granted is roughly the following (references indicate the level): algebra, D. S. Dummit and R. M. Foote [31]; functional analysis, W. Rudin [86]; theory of differentiable manifolds, L. W. Tu [91]; differential geometry, S. Kobayashi and K. Nomizu [63]; Lie theory, F. W. Warner [92].

# 2

## REVIEW OF THE THEORY OF COMPACT LIE GROUPS

WE review some basic notions surrounding the Laplacian on a compact Lie group that is endowed with a left-invariant metric. This will also serve as an opportunity to introduce the notation we use throughout this dissertation. For basic Lie theory, we refer to [92, Ch. 3]. For background in differential geometry, see [22, Ch. 1]; the same reference contains (Ch. 3) an excellent account on the differential geometry of homogeneous spaces. For general reference on the representation theory of (locally) compact groups, we refer to [83].

### 2.1 ANALYTICAL ASPECTS

2.1.1 Throughout this dissertation,  $G$  denotes a compact Lie group unless mentioned otherwise. We shall always denote the identity element of  $G$  by  $e$ . We denote by  $\ell_g$  and  $r_g$ , respectively, the left and right translations on  $G$  by  $g \in G$ , that is,

$$\begin{aligned}\ell_g : G &\rightarrow G, \\ x &\mapsto gx, \\ r_g : G &\rightarrow G, \\ x &\mapsto xg.\end{aligned}$$

A vector field  $\tilde{X}$  on  $G$  is *left-invariant* if

$$\ell_{g*}\tilde{X} = \tilde{X}$$

for all  $g$  in  $G$ . Here  $\ell_{g*}$  denotes the differential of  $\ell_g$ . A left-invariant differential form is defined similarly using the pullback homomorphism  $\ell_g^*$  on the space  $\Omega^*(G)$  of differential forms on  $G$ .

If  $\tilde{X}$  is a left-invariant vector field, then it is completely deter-

mined by its value at the identity, since

$$\tilde{X}_g = \ell_{g*} \tilde{X}_e.$$

Conversely, a tangent vector  $X$  at the identity induces a left-invariant vector field  $\tilde{X}$  on  $G$  by defining the value of  $\tilde{X}$  at  $g$  by

$$\tilde{X}_g = \ell_{g*} X.$$

Therefore, there is a one-to-one correspondence between the space  $\mathfrak{X}(G)^G$  of left-invariant vector fields and the tangent space  $T_e G$  of  $G$  at the identity. The space  $\mathfrak{X}(G)^G$  is closed under the commutator [92, Prop. 3.7(c), p. 84]. Defining the bracket operation on  $T_e G$  as

$$[X, Y] := [\tilde{X}, \tilde{Y}]_e$$

makes  $T_e G$  a Lie algebra. This is known as the *Lie algebra of  $G$* , and it is customary to denote it by the lowercase black-letter  $\mathfrak{g}$ .

For each  $g$  in  $G$ , let  $c(g)$  be the conjugation map  $G \rightarrow G$ ,  $h \mapsto ghg^{-1}$ . Mapping  $g$  to the differential of  $c(g)$  at the identity gives the *adjoint representation of  $G$* :

$$\begin{aligned} \text{Ad} : G &\rightarrow \text{Aut}(\mathfrak{g}), \\ g &\mapsto c(g)_{*,e}. \end{aligned} \tag{2.1.2}$$

We shall denote the image of  $g \in G$  under  $\text{Ad}$  by  $\text{Ad}_g$ . The differential of  $\text{Ad}$  at the identity, in turn, gives the *adjoint representation of  $\mathfrak{g}$* :

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \text{End}(\mathfrak{g}), \\ X &\mapsto \text{Ad}_{*,e}(X). \end{aligned}$$

We shall denote the image of  $X \in \mathfrak{g}$  under  $\text{ad}$  by  $\text{ad}_X$ . It is well-known that

$$\text{ad}_X(Y) = [X, Y]$$

for  $X$  and  $Y$  in  $\mathfrak{g}$ . (See, for instance, [92, Prop. 3.47].)

**2.1.3 BI-INVARIANT METRICS.** Suppose  $\langle \cdot, \cdot \rangle$  is an inner product on  $\mathfrak{g}$ . Extend this inner product to the whole tangent bundle  $TG$  by left translations; that is, for  $X$  and  $Y$  in  $T_g G$ , set

$$\langle X, Y \rangle = \langle \ell_{g*}^{-1} X, \ell_{g*}^{-1} Y \rangle.$$

This defines a left-invariant metric on  $G$  in the sense that

$$\langle X, Y \rangle = \langle \ell_{x*} X, \ell_{x*} Y \rangle$$

for any  $x$  in  $G$ . This metric would also be right-invariant if and only if the inner product on  $\mathfrak{g}$  is  $\text{Ad}(G)$ -invariant; this owes to the following equalities: For  $X$  and  $Y$  in  $\mathfrak{g}$ ,

$$\langle r_{g*} X, r_{g*} Y \rangle = \langle \ell_{g^{-1}*} \circ r_{g*} X, \ell_{g^{-1}*} \circ r_{g*} Y \rangle = \langle \text{Ad}_{g^{-1}} X, \text{Ad}_{g^{-1}} Y \rangle.$$

So we see that there is a one-to-one correspondence between the  $\text{Ad}(G)$ -invariant inner products on  $\mathfrak{g}$  and the bi-invariant (that is, left- and right-invariant) metrics on  $G$ . Henceforth, we shall say

that an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  is *invariant* if the  $\text{Ad}(G)$ -action is symmetric or the  $\text{ad}(\mathfrak{g})$ -action is antisymmetric with respect to  $\langle \cdot, \cdot \rangle$ .

For compact Lie groups, we can always make an inner product on  $\mathfrak{g}$  invariant by replacing (if necessary) the original inner product with its average over  $G$ :

$$\langle\langle X, Y \rangle\rangle := \int_G \langle \text{Ad}_g X, \text{Ad}_g Y \rangle dg.$$

Here  $dg$  is a Haar measure of our choice on  $G$ .

From now on we shall always assume that our compact Lie group  $G$  is endowed with a bi-invariant metric  $\langle \cdot, \cdot \rangle$ .

**2.1.4 THE KILLING FORM.** There is a natural bilinear form on the Lie algebra  $\mathfrak{g}$ , namely, the *Killing form*,

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathbb{C}, \\ (X, Y) &\mapsto \text{tr}(\text{ad}_X \circ \text{ad}_Y). \end{aligned}$$

Owing to the fact that  $\text{ad}_{\text{Ad}_g(X)} = \text{Ad}_g \circ \text{ad}_X \circ \text{Ad}_{g^{-1}}$ , we have the symmetry

$$\kappa(\text{Ad}_g X, Y) = \kappa(X, \text{Ad}_{g^{-1}} Y).$$

Differentiating both sides of this equation with respect to  $g$  gives

$$\kappa(\text{ad}_Z X, Y) = -\kappa(X, \text{ad}_Z Y).$$

Lie groups (not necessarily compact) whose Killing form is non-degenerate are precisely the *semisimple* ones. If  $G$  is compact and semisimple, then the Killing form is negative definite (for a proof, see [54, Prop. 6.6]). So, for compact semisimple Lie groups, we may use  $-\kappa(\cdot, \cdot) = \langle \cdot, \cdot \rangle_\kappa$  as the inner product for  $\mathfrak{g}$ . This is an invariant inner product, hence, yielding a bi-invariant metric on  $G$ .

**2.1.5 RIEMANNIAN VOLUME FORM.** Let  $X_1, \dots, X_n$  be a orthonormal basis for  $\mathfrak{g}$ . Let  $\theta^1, \dots, \theta^n$  be the dual basis for  $\mathfrak{g}^*$ . Extending the form

$$\theta_1 \wedge \dots \wedge \theta^n \in \wedge^n \mathfrak{g}^*$$

to all of  $G$  by left translations, we obtain a bi-invariant volume form on  $G$ . This is identical to the Riemannian<sup>1</sup> volume form of  $G$ .

Every left-invariant volume form on  $G$  is completely determined by its value at the identity; so every left-invariant volume form on  $G$  is a scalar multiple of the Riemannian volume form, and hence, they are all bi-invariant. Henceforth, we shall speak of an *invariant volume form* without reference to left or right invariance.

<sup>1</sup> Suppose  $M$  is a Riemannian manifold with metric  $\langle \cdot, \cdot \rangle$ . For a local coordinate system  $(x_1, \dots, x_n)$ , let  $\eta$  denote the matrix whose  $(i, j)$ -entry is  $\eta_{ij} = \langle \partial_i, \partial_j \rangle$  where  $\partial_i = \partial/\partial x_i$ . Let  $\eta^{ij}$  denote the  $(i, j)$ -entry of  $\eta^{-1}$ . Let  $\text{vol} := |\det(\eta)|^{1/2} dx^1 \wedge \dots \wedge dx^n$ , where  $\{dx^1, \dots, dx^n\}$  is the dual basis for the cotangent space with respect to  $\partial_1, \dots, \partial_n$ . This local expression for  $\text{vol}$  defines a global  $n$ -form called the *Riemannian volume form* of  $M$ .

A similar statement can be made for invariant measures. Let  $C(G)$  be the space of continuous functions on  $G$ . A *left Haar measure* on  $G$  is a positive linear functional  $\Upsilon : C(G) \rightarrow \mathbb{R}$  that is left invariant in the sense that  $\Upsilon(f) = \Upsilon(\ell_g^* f)$  for all  $g$  in  $G$ , where  $\ell_g^* f$  is the pullback of  $f$  along the left translation  $\ell_g$ . A *right Haar measure* is defined similarly using the right translations. An invariant volume defines a left Haar measure that is also a right Haar measure. Because a left Haar measure on  $G$  is unique up to a constant factor (see [61, Thm. 6.8]), there is a one-to-one correspondence between invariant volume forms and left Haar measures on  $G$ . Thus, every left Haar measure on  $G$  must be a right Haar measure, which is to say that  $G$  is *unimodular*.

**2.1.6 THE LAPLACIAN.** The Laplacian is defined as follows. (The definition is appropriate for any Riemannian manifold.) Let  $C^\infty(G)$  be the space of smooth functions on  $G$ , and let  $\mathfrak{X}(G)$  denote the space of smooth vector fields on  $G$ . We define the *gradient* operator  $\text{grad} : C^\infty(G) \rightarrow \mathfrak{X}(G)$  and the *divergence* operator  $\text{div} : \mathfrak{X}(G) \rightarrow C^\infty(G)$  by

$$\begin{aligned}\langle \text{grad } f, X \rangle &= Xf, \\ (\text{div } X) \text{vol} &= L_X \text{vol},\end{aligned}$$

where  $\text{vol}$  is the Riemannian volume form on  $G$  and  $L_X$  is the Lie derivative with respect to  $X$ . The *Laplacian* (or the *Laplace-Beltrami operator*)  $\Delta_G : C^\infty(G) \rightarrow C^\infty(G)$  is defined by

$$\Delta_G f := \text{div}(\text{grad } f).$$

The definition of  $\Delta_G$  depends only on the metric. Because our metric  $\langle \cdot, \cdot \rangle$  is bi-invariant, so is the Laplacian in the sense that  $\Delta_G$  commutes with  $\ell_g^*$  and  $r_g^*$  for all  $g$  in  $G$ .

An expression for the Laplacian in local coordinates  $(x_1, \dots, x_n)$  can be given as follows. (See [55, Ch. II, § 2] for details.) Let  $\eta$  be the matrix defined by  $\eta_{ij} = \langle \partial_i, \partial_j \rangle$  where  $\partial_i = \partial/\partial x_i$ . Let  $\eta^{ij}$  denote the  $(i, j)$ th entry of  $\eta^{-1}$ . Then,

$$\Delta_G f = \frac{1}{|\det(\eta)|^{1/2}} \sum_{i,j} \partial_i (|\det(\eta)|^{1/2} \eta^{ij} \partial_j f). \quad (2.1.7)$$

**2.1.8 THE SPECTRUM OF THE LAPLACIAN.** The Laplacian  $\Delta_G$  is defined over  $C^\infty(G)$ , which is a dense subspace of  $L^2(G)$ . The domain can be extended to the Sobolev space  $H^2(G)$ , that is, the space of  $L^2$ -functions  $u$  on  $G$  such that all distributional 1st and 2nd derivatives lie in  $L^2(G)$ . This extension, which we denote by  $\bar{\Delta}_G : H^2(G) \rightarrow L^2(G)$ , is the unique self-adjoint extension of  $\Delta_G : C^\infty(G) \rightarrow L^2(G)$ . Thus, in the language of unbounded operator theory, the Laplacian is *essentially self-adjoint* on  $C^\infty(G)$ . For details on these matters, see [85, Ch. 5].

Let  $1$  denote the inclusion map  $H^2(G) \hookrightarrow L^2(G)$ . The operator  $1 - \Delta_G$  admits an inverse that is compact (see [89, § 5.1]). So, by the Spectral theorem, the eigenfunctions  $\{u_k\}_{k=1}^\infty$  of  $(1 - \Delta_G)^{-1}$  form an orthonormal basis for  $L^2(G)$ . They are also eigenfunctions of  $-\Delta_G$ . Owing to the regularity of elliptic differential operators, the eigenfunctions are of class  $C^\infty$ . We can also conclude from the Spectral theorem that the eigenfunctions  $u_k$  can be ordered in such a way that the corresponding eigenvalues  $-\lambda_k$  of  $\Delta_G$  form a nonincreasing unbounded sequence of negative real numbers,

$$0 > -\lambda_1 \geq -\lambda_2 \geq -\lambda_3 \geq \dots$$

**2.1.9 THE HEAT DIFFUSION OPERATOR.** The *heat diffusion operator* of  $\Delta_G$  is defined as

$$e^{t\Delta_G} := \begin{pmatrix} e^{-t\lambda_1} & & & \\ & e^{-t\lambda_2} & & \\ & & e^{-t\lambda_3} & \\ & & & \ddots \end{pmatrix}$$

for  $t \geq 0$ , where the matrix on the right-hand side is with respect to the basis consisting of the eigenfunctions of the Laplacian.

The heat diffusion operator is an integral operator with a  $C^\infty$ -kernel. (This owes to the Schwartz kernel theorem; see [89, Ch. 4, § 6, p. 345] or [85, Prop. 5.31, p. 83].) This means that there is some  $C^\infty$ -function  $(t, x, y) \mapsto K_t(x, y)$  on  $]0, \infty[ \times G \times G$  such that

$$(e^{t\Delta_G} f)(x) = \int_G K_t(x, y) f(y) \text{vol}_y \quad (2.1.10)$$

for any  $f$  in  $L^2(G)$ ; here  $\text{vol}_y$  is the Riemannian volume form at  $y$ . The kernel  $K_t$  is called the *heat kernel* of  $\Delta_G$ . It has the following fundamental properties:

- (i) It satisfies the heat equation

$$(\partial_t - \Delta_G)K_t(x, y) = 0,$$

where the Laplacian applies to the first variable  $x$ .

- (ii) As  $t \rightarrow 0+$ , it converges to the Dirac delta distribution  $\delta(x - y)$  in the sense that, for any smooth function  $f$  on  $G$ ,

$$\int_G K_t(x, y) f(y) \text{vol}_y \rightarrow f(x)$$

uniformly.

The smooth kernel  $K_t$  is uniquely determined by the above two properties; see [85, Prop. 7.5, p. 96].

Recall that  $\Delta_G$  is bi-invariant; in particular, we have

$$\Delta_G = \ell_g^* \circ \Delta_G \circ \ell_{g^{-1}}^*$$

for any  $g$  in  $G$ , and thus,

$$(e^{t\Delta_G} f) = \ell_g^* \circ e^{t\Delta_G} \circ \ell_{g^{-1}}^*.$$

This means, in terms of the heat kernel,

$$\int_G K_t(x, y) f(y) \operatorname{vol}_y = \int_G K_t(gx, y) f(g^{-1}y) \operatorname{vol}_y$$

for any  $f$  in  $L^2(G)$ . Substituting  $gy$  for  $y$  in the right-hand side and using the invariance of the volume form, this equation can be restated as

$$\int_G K_t(x, y) f(y) \operatorname{vol}_y = \int_G K_t(gx, gy) f(y) \operatorname{vol}_y.$$

This implies

$$K_t(x, y) = K_t(gx, gy).$$

Substituting  $g$  with  $x^{-1}$  gives us

$$K_t(x, y) = K_t(e, x^{-1}y).$$

Hence, the heat kernel is completely determined by the function

$$k_t : y \mapsto K_t(e, y). \quad (2.1.11)$$

We shall call this the *heat convolution kernel* of  $\Delta_G$ . With it, Equation 2.1.10 can be rephrased as

$$(e^{t\Delta_G} f)(x) = \int_G k_t(x^{-1}y) f(y) \operatorname{vol}_y. \quad (2.1.12)$$

Substituting  $xy$  for  $y$  and using the invariance of the volume form, we get

$$(e^{t\Delta_G} f)(x) = \int_G k_t(y) f(xy) \operatorname{vol}_y. \quad (2.1.13)$$

A consequence of  $e^{t\Delta_G}$  being an integral operator with a  $C^\infty$ -kernel is that it has a finite trace. The trace of the heat diffusion operator,  $Z(t) = \operatorname{tr}(e^{t\Delta_G})$ , is called the *partition function* or the *heat-trace* of  $\Delta_G$ . It can be calculated in terms of the heat kernel as follows (see [85, Thm. 8.10, p. 113]):

$$Z(t) = \int_G K_t(x, x) \operatorname{vol}_x = k_t(e) \operatorname{vol}(G). \quad (2.1.14)$$

Since a Riemannian manifold is locally “flat”, it is reasonable to expect that the heat convolution kernel, which governs the heat diffusion, looks almost Gaussian for small time  $t$  near  $e$ . We seek a formal solution of the form

$$\tilde{k}_t(x) := h_t(x) (a_0(x) + a_1(x)t + a_2(x)t^2 + \cdots), \quad (2.1.15)$$

where  $h_t$  is the Gaussian kernel:

$$h_t(x) = \frac{e^{-d(x)^2/4t}}{(4\pi t)^{\dim M/2}}. \quad (2.1.16)$$

Here  $d(x)$  denotes the distance from  $e$  to  $x$ . Write  $\tilde{k}_t = h_t s_t$  where  $s_t := \sum_{i=0}^{\infty} a_i t^i$ . Then, on a neighborhood  $U$  of  $e$  where  $\log : U \rightarrow \mathfrak{g}$ ,



$x \mapsto \exp^{-1}(x)$ , is well-defined, the heat equation  $(\partial_t + \Delta_G)\tilde{k}_t = 0$  can be rewritten as (see [85, Eq. 7.16, p. 102] or [12, Prop. 2.24, p. 82])

$$h_t \left( \partial_t + \frac{1}{t} \tilde{R} + \tilde{j} \circ \Delta_G \circ \frac{1}{\tilde{j}} \right) s_t = 0, \quad (2.1.17)$$

where  $\tilde{R}$  is the radial vector field in normal coordinates (this means that the pushforward of  $\tilde{R}_x$  along  $\log$  is equal to  $X := \log(x)$  as we identify the tangent space  $T_x \mathfrak{g}$  with  $\mathfrak{g}$ ) and  $\tilde{j} = j \circ \log$  with

$$j(X) = \det^{1/2} \left( \frac{\sinh \operatorname{ad}_X / 2}{\operatorname{ad}_X / 2} \right).$$

Setting each coefficients of powers of  $t$  in Equation 2.1.17 as 0, we obtain a family of differential equations:

$$\begin{aligned} \tilde{R}a_0 &= 0, \\ (\tilde{R} + n)a_i &= -\tilde{j}\Delta_G \left( \frac{a_{i-1}}{\tilde{j}} \right), \quad i \geq 1. \end{aligned}$$

This can be solved inductively under the condition  $a_0(e) = 1$ . The formal power series 2.1.15 is called the *asymptotic heat kernel* or the *asymptotic expansion* for the true heat kernel  $k_t$  because of the following property: Let  $r$  be any nonnegative integer, and let  $\|\cdot\|_{C^r}$  denote the usual norm<sup>2</sup> on the space  $C^r(G)$  of  $C^r$ -functions on  $G$ ; then, for each nonnegative integer  $n$  there is a positive integer  $N$  such that, for each  $m \geq N$ ,

$$\left\| k_t - h_t \sum_{i=0}^m a_i t^i \right\|_{C^r} \leq C|t|^n$$

for sufficiently small  $t$ ; here  $C$  is a constant that depends on  $n$ ,  $m$ , and  $r$ . For details, see [85, Thm. 7.15, p. 101] or [12, Thm. 2.26, p. 83, Thm. 2.29, p. 85].

2.1.18 In general, a function  $f : ]0, \infty[ \rightarrow E$ , where  $E$  is a Banach space, is said to be *asymptotically equal* to the formal sum  $\sum_{i=0}^{\infty} a_i$  of functions  $a_i : ]0, \infty[ \rightarrow E$  if, for each positive integer  $n$  there is a positive integer  $N$  such that

$$f(t) - \sum_{i=0}^m a_i(t) = O(t^n)$$

holds for all  $m \geq N$ . (The above equation means that there is some constant  $C$  such that  $\|f(t) - \sum_{i=0}^m a_i(t)\| \leq C|t|^n$  for sufficiently small

<sup>2</sup> Let  $d = \dim M$  and let  $\alpha = (\alpha_1, \dots, \alpha_d)$  denote a multi-index in  $\mathbb{N}^d$ . Then the standard  $C^r$ -norm of  $f$  in  $C^r(M)$  is  $\|f\|_{C^r} = \sup_{|\alpha| \leq r} \|\partial^\alpha f\|_{\sup}$  where  $|\alpha| := \sum_{i=1}^d \alpha_i$ ,  $\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_d}}{\partial x_d}$ , and  $\|f\|_{\sup} := \sup_{x \in M} |f(x)|$ .

t.) If this is the case, we write

$$f \sim \sum_{i=0}^{\infty} a_i,$$

and call  $\sum_{i=0}^{\infty} a_i$  an *asymptotic expansion* for  $f$ . In the special case where  $f \sim 0$ , that is,  $f(t) = O(t^n)$  for all positive integer  $n$ , we shall say that  $f$  is of *order*  $t^\infty$  or write  $f = O(t^\infty)$ . We say that  $g : ]0, \infty[ \rightarrow E$  is *asymptotically equal* to  $f$  and write

$$f \sim g,$$

if  $f - g$  is asymptotically equal to 0.

## 2.2 ALGEBRAIC ASPECTS

2.2.1 We pointed out earlier that the Laplacian is bi-invariant in that it commutes with  $\ell_g^*$  and  $r_g^*$  for all  $g \in G$ . In general, a differential operator  $D$  on  $C^\infty(G)$  is said to be (*left*) *invariant* if

$$D = \ell_g^* \circ D \circ \ell_{g^{-1}}^*$$

for all  $g$  in  $G$ . It is said to be *bi-invariant* if it furthermore satisfies

$$D = r_g^* \circ D \circ r_{g^{-1}}^*.$$

We shall denote by  $D(G)$  the algebra of invariant differential operators on  $C^\infty(G)$ . Assuming  $G$  is connected, the subalgebra of bi-invariant operators is precisely the center of  $D(G)$ . But before we review why this is so, let us briefly go over the terminologies and notations of the representation theory that we shall use.

2.2.2 A *representation* of a compact Lie group  $G$  is a continuous group homomorphism  $u : G \rightarrow \text{Aut}(V)$  where  $V$  is a topological vector space. (If  $V$  is finite-dimensional, then a continuous representation is automatically of class  $C^\infty$  because every continuous group homomorphism of Lie groups are of class  $C^\infty$ ; see [92, Thm. 3.39, p. 109].) Most of the time, we shall be interested in the *unitary representations* of  $G$ ; that means the vector space  $V$ , called the *representation space* of  $u$ , is a Hilbert space and that the representation  $u$  maps  $G$  into the space of unitary operators on  $V$ . The unitarity of the representation is equivalent to the invariance of the inner product under the action of  $G$ ; this owes to the following equality:

$$\langle u(g)v | u(g)w \rangle = \langle u(g)^* u(g)v | w \rangle,$$

for  $g \in G$  and  $(v, w) \in V \times V$ . It turns out that any representation of a compact Lie group on a Hilbert space is equivalent to a unitary representation; indeed, for any inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , the averaged inner product  $\langle\langle v, w \rangle\rangle := \int_G \langle u(g)v, u(g)w \rangle dg$  is an invariant inner product on the same Hilbert space, and the norm associated to the new inner product is equivalent to the original one (see

[83, Prop. 2.2, p. 14]).

If we have a representation  $u : G \rightarrow \text{Aut}(V)$ , then we say that  $V$  is a  $G$ -vector space. The dimension of  $V$  is referred to as the *dimension* of the representation. The representation is said to be *real* or *complex* according to the field of scalars of  $V$ . For our purposes, we shall mainly consider the complex representations.

2.2.3 NOTATION. Let  $u : G \rightarrow \text{Aut}(V)$  be a representation. For  $g \in G$  and  $v \in V$ , we shall often write  $u(g)(v)$  as  $u_g(v)$ , or even as  $g \cdot v$  if there is no risk of confusion.

2.2.4 A linear map  $T : V \rightarrow W$  between two  $G$ -vector spaces is a  $G$ -map (or an *intertwiner*) if it is  $G$ -equivariant, that is,  $g \cdot T(v) = T(g \cdot v)$  holds for all  $g \in G$  and all  $v \in V$ . The space of  $G$ -maps  $V \rightarrow W$  is denoted by  $\text{Hom}_G(V, W)$ . If the  $G$ -map  $T : V \rightarrow W$  is homeomorphic, then we say that  $V$  and  $W$  are  $G$ -isomorphic and that the representations on  $V$  and  $W$  are *equivalent*.

2.2.5 Let  $u : G \rightarrow \text{Aut}(V)$  be a finite-dimensional representation. Its *character* is the function

$$\begin{aligned} \chi_u : G &\rightarrow \mathbb{k}, \\ g &\mapsto \text{tr}(u_g), \end{aligned}$$

where  $\mathbb{k}$  is the field of scalars of  $V$ .

The *induced Lie algebra representation*  $u_* : \mathfrak{g} \rightarrow \text{End}(V)$  associated to  $u$  is defined as the differential of  $u$  at the identity. This means that, for  $X$  in  $\mathfrak{g}$ ,

$$u_*(X)(v) = \left. \frac{d}{dt} \right|_0 u(\exp tX)(v).$$

Put in another way,

$$e^{u_*(X)} = u(\exp(X)); \quad (2.2.6)$$

see [92, Thm. 3.32, p. 104].

The *dual* (or *contragredient*) representation induced by  $u$  is the representation  $\check{u} : G \rightarrow \text{Aut}(V^*)$  defined by

$$(\check{u}_g \phi)(v) = \phi(u_g^{-1}(v)) \quad (2.2.7)$$

where  $\phi \in V^*$  and  $v \in V$ .

If  $u' : G \rightarrow \text{Aut}(V')$  is another finite-dimensional representation, then the *direct sum* and the *tensor product* representation  $u \oplus u' : G \rightarrow \text{Aut}(V \oplus V')$  and  $u \otimes u' : G \rightarrow \text{Aut}(V \otimes V')$ , respectively, are defined by

$$\begin{aligned} (u \oplus u')_g(v \oplus v') &= u_g v \oplus u'_g v', \\ (u \otimes u')_g(v \otimes v') &= u_g v \otimes u'_g v'. \end{aligned}$$

2.2.8 SCHUR'S LEMMA. A simple yet very useful theorem in representation theory is *Schur's lemma*. There are several versions of

Schur's lemma, one of which is as follows: Let  $V$  and  $W$  be finite-dimensional vector spaces over  $\mathbb{C}$  (or any algebraically closed field). Suppose there is a family of pairs of linear maps,

$$\{(\phi_a, \psi_a) \in \text{End}(V) \times \text{End}(W)\}_{a \in A},$$

with respect to which the spaces  $V$  and  $W$  are *irreducible*, that is, they admit no proper nonzero subspaces that are invariant under  $\{\phi_a\}_{a \in A}$  and  $\{\psi_a\}_{a \in A}$ , respectively. Then the space of intertwiners,

$$\{T : V \rightarrow W \mid T \circ \phi_a = \psi_a \circ T, \forall a \in A\},$$

is 1-dimensional or 0-dimensional according to whether  $\dim V$  is equal to  $\dim W$ . For a proof, see [99, Lem. 3.1.C, p. 83].

**2.2.9 IRREDUCIBILITY.** Let  $u : G \rightarrow \text{Aut}(V)$  be a representation. A subspace  $W$  of  $V$  is said to be *invariant* if  $u_g(W) \subseteq W$  for all  $g$  in  $G$ . If  $V$  admits a closed invariant proper subspace other than  $\{0\}$ , then we say that the representation is *reducible*; otherwise, the representation is said to be *irreducible*. A reducible representation is said to be *completely reducible* if the representation space is isomorphic to a direct sum of irreducible subspaces (that is, a Hilbert space direct sum, if the representation is on a Hilbert space).

Suppose the representation  $u$  is a unitary representation on a Hilbert space. If  $W$  is an invariant subspace, then its orthogonal complement  $W^\perp$  is also invariant, owing to the invariance of the inner product. Thus, any representation of a compact Lie group on a finite-dimensional space or an infinite-dimensional Hilbert space, is reducible if and only if it is completely reducible (by Zorn's lemma).

It turns out that an irreducible representation of a compact Lie group on a complex Hilbert space must be finite-dimensional (see [18, Thm. 4.3, p. 25]). Hence, any complex representation of a compact Lie group on a Hilbert space is a direct sum of finite-dimensional irreducible subspaces.

The complete set of distinct (that is, non-equivalent) irreducible unitary representations of  $G$  is called the *unitary dual* of  $G$ ; we shall denote it by  $\widehat{G}$ .

For each  $u$  in  $\widehat{G}$ , the *u-isotypic component* of a  $G$ -vector space  $V$  is defined as the maximal subspace  $W$  of  $V$  on which the  $G$ -action is isomorphic to a direct sum of copies of  $u$ ; owing to Schur's lemma, we have

$$W \simeq \text{Hom}_G(U, V) \otimes U,$$

where  $U$  is the representation space of  $u$ . The dimension of the space  $\text{Hom}_G(U, V)$  is called the *multiplicity* of  $u$  in  $V$ .

**2.2.10 THE UNIVERSAL ENVELOPING ALGEBRA.** The left translations on  $G$  induces a natural  $G$ -action on  $C^\infty(G)$ ; the *left-regular action* of

$g \in G$  on  $f \in C^\infty(G)$  is defined by

$$(L(g)f)(x) = f(g^{-1}x).$$

There is also the *right-regular action* induced by the right translations:

$$(R(g)f)(x) = f(xg).$$

The induced Lie algebra representation of  $X$  in  $\mathfrak{g}$  associated to the right-regular representation is given by

$$(R_*(X)f)(x) = \left. \frac{d}{dt} \right|_0 f(x \exp(tX)).$$

Note that this action is exactly that of the left-invariant vector field  $\tilde{X}$  on  $G$  generated by  $X$ . Thus,

$$(R_*(X)f)(x) = (\tilde{X}f)(x).$$

Hence, we have a map

$$\begin{aligned} \tau: \mathfrak{g} &\rightarrow D(G), \\ X &\mapsto \tilde{X}. \end{aligned}$$

Because  $[\widetilde{[X, Y]}] = [\tilde{X}, \tilde{Y}]$  (where the bracket on the right is just the commutator in  $D(G)$ ) the map  $\tau$  extends to the *universal enveloping algebra*  $\mathcal{U}(\mathfrak{g})$  of  $\mathfrak{g}$ , which is the algebra constructed by first taking the tensor algebra  $T(\mathfrak{g})$  of  $\mathfrak{g}$  and then taking the quotient by the ideal  $J(\mathfrak{g})$  generated by the elements of the form  $X \otimes Y - Y \otimes X - [X, Y]$ . The extended map

$$\begin{aligned} \tau: \mathcal{U}(\mathfrak{g}) &\rightarrow D(G), \\ X_1 \otimes \cdots \otimes X_k + J(\mathfrak{g}) &\mapsto \tilde{X}_1 \circ \cdots \circ \tilde{X}_k, \end{aligned} \quad (2.2.11)$$

is an algebra isomorphism (see [54, Ch. II, Prop. 1.9]). It is customary to write  $X_1 \otimes \cdots \otimes X_k + J(\mathfrak{g})$  in  $\mathcal{U}(\mathfrak{g})$  as  $X_1 \cdots X_k$  and  $\tilde{X}_1 \circ \cdots \circ \tilde{X}_k$  in  $D(G)$  as  $\tilde{X}_1 \cdots \tilde{X}_k$ .

Let us verify the claim we made in Section 2.2.1 that, if  $G$  is connected, the center of  $D(G)$  is the subalgebra of bi-invariant differential operators. The argument we give here is from [55, p. 283]. We need to show that an invariant differential operator  $D$  commutes with every operator in  $D(G)$  if and only if  $r_g^* \circ D \circ r_{g^{-1}}^* = D$  for all  $g$  in  $G$ . As we can see from the isomorphism 2.2.11, the elements of the form  $\tilde{X}_1 \cdots \tilde{X}_k$  span  $D(G)$ ; hence, we may assume that  $D = \tilde{X}_1 \cdots \tilde{X}_k$ . Moreover, the left-invariant vector fields generate  $D(G)$ ; so it is sufficient to check that the commutator  $[\tilde{Y}, D] := \tilde{Y}D - D\tilde{Y}$  is zero for any  $Y$  in  $\mathfrak{g}$  if and only if  $r_g^* \circ D \circ r_{g^{-1}}^* = D$  for all  $g$  in  $G$ . For the “only if” assertion, start by noting that

$$(r_g^* \circ \tilde{X} \circ r_{g^{-1}}^* f)(x) = \left. \frac{d}{dt} \right|_0 f(xg \exp(tX)g^{-1}) = \left. \frac{d}{dt} \right|_0 f(x \exp(t \operatorname{Ad}_g X))$$

for a smooth function  $f$  on  $G$ . Hence,

$$r_g^* \circ \tilde{X} \circ r_{g^{-1}}^* = \widetilde{\text{Ad}_g(X)}. \quad (2.2.12)$$

If  $g = \exp(Y)$ , then  $\text{Ad}_g(X) = e^{\text{ad}_Y}(X)$ . So

$$\widetilde{\text{Ad}_g(X)} = e^{\text{ad}_{\tilde{Y}}}(\tilde{X}),$$

where  $\text{ad}_{\tilde{Y}} : \mathfrak{X}(G)^G \rightarrow \mathfrak{X}(G)^G$ ,  $\tilde{X} \mapsto [\tilde{Y}, \tilde{X}]$ , and  $e^{\text{ad}_{\tilde{Y}}} = \sum_{n=0}^{\infty} \text{ad}_{\tilde{Y}}^n / n!$  (the convergence is not an issue since  $\mathfrak{X}(G)^G$  is finite-dimensional). The domain of  $\text{ad}_{\tilde{Y}}$  extends, as an inner derivation, to  $D(G)$ . Then, we have

$$\text{ad}_{\tilde{Y}}(D) = [\tilde{Y}, D].$$

Note that  $\text{ad}_{\tilde{Y}}(D)$  is of order at most  $k$ . So the set  $\{\text{ad}_{\tilde{Y}}^n(D) : n \in \mathbb{N}\}$  is contained in the finite-dimensional subspace of  $D(G)$  consisting of operators of order at most  $k$ , and hence,  $e^{\text{ad}_{\tilde{Y}}}(D) := \sum_{n=0}^{\infty} \text{ad}_{\tilde{Y}}^n(D)/n!$  is convergent. Now

$$\begin{aligned} r_g^* \circ (\tilde{X}_1 \cdots \tilde{X}_k) \circ r_{g^{-1}}^* &= (r_g^* \circ \tilde{X}_1 \circ r_{g^{-1}}^*) \circ \cdots \circ (r_g^* \circ \tilde{X}_k \circ r_{g^{-1}}^*) \\ &= \widetilde{\text{Ad}_g(X_1)} \cdots \widetilde{\text{Ad}_g(X_k)}, \end{aligned}$$

where we have used Equation 2.2.12 for the last equality. So, for  $g = \exp(Y)$ ,

$$\begin{aligned} r_g^* \circ (\tilde{X}_1 \cdots \tilde{X}_k) \circ r_{g^{-1}}^* &= e^{\text{ad}_{\tilde{Y}}}(\tilde{X}_1) \cdots e^{\text{ad}_{\tilde{Y}}}(\tilde{X}_k) \\ &= \left( \sum_{n_1=0}^{\infty} \frac{\text{ad}_{\tilde{Y}}^{n_1}(\tilde{X}_1)}{n_1!} \right) \cdots \left( \sum_{n_k=0}^{\infty} \frac{\text{ad}_{\tilde{Y}}^{n_k}(\tilde{X}_k)}{n_k!} \right) \\ &= \sum_{N=0}^{\infty} \left( \sum_{n_1+\cdots+n_k=N} \frac{\text{ad}_{\tilde{Y}}^{n_1}(\tilde{X}_1) \cdots \text{ad}_{\tilde{Y}}^{n_k}(\tilde{X}_k)}{n_1! \cdots n_k!} \right). \end{aligned} \quad (2.2.13)$$

Since  $\text{ad}_{\tilde{Y}}$  is a derivation, we have.

$$\text{ad}_{\tilde{Y}}^N(\tilde{X}_1 \cdots \tilde{X}_k) = \sum_{n_1+\cdots+n_k=N} \frac{N!}{n_1! \cdots n_k!} \text{ad}_{\tilde{Y}}^{n_1}(\tilde{X}_1) \cdots \text{ad}_{\tilde{Y}}^{n_k}(\tilde{X}_k).$$

Thus, Equation 2.2.13 can be rewritten as

$$r_g^* \circ (\tilde{X}_1 \cdots \tilde{X}_k) \circ r_{g^{-1}}^* = \sum_{N=0}^{\infty} \frac{\text{ad}_{\tilde{Y}}^N(\tilde{X}_1 \cdots \tilde{X}_k)}{N!} = e^{\text{ad}_{\tilde{Y}}}(\tilde{X}_1 \cdots \tilde{X}_k).$$

Therefore, if  $[\tilde{Y}, D] = 0$  for all  $Y$  in  $\mathfrak{g}$ , then  $r_g^* \circ D \circ r_{g^{-1}}^* = D$  for all  $g$  in the image of the exponential map. But, since  $G$  is connected by hypothesis, we may conclude that  $r_g^* \circ D \circ r_{g^{-1}}^* = D$  holds for all  $g$  in  $G$ ; this is because<sup>3</sup>, for a neighborhood  $U$  of  $e$  that is in the

<sup>3</sup> We could have, at this point, referred to the fact that the exponential map of a compact connected Lie group is surjective (see [18, Thm. 16.3, p. 103]); but the argument presented here works for any connected Lie group.

image of the exponential map, any element of  $G$  can be expressed as a product of finite number of elements in  $U$  (see [92, Thm. 3.18, p.93]). Conversely, suppose  $D$  is bi-invariant; then

$$r_g^* \circ D = D \circ r_g^*$$

for all  $g$  in  $G$ . So, for a smooth function  $f$  on  $G$ ,

$$(r_g^*(\tilde{X}_1 \cdots \tilde{X}_k f))(x) = (\tilde{X}_1 \cdots \tilde{X}_k(r_g^* f))(x).$$

This means that

$$\begin{aligned} \frac{d}{dt_1} \Big|_0 \cdots \frac{d}{dt_k} \Big|_0 f(xg \exp(t_1 X_1) \cdots \exp(t_k X_k)) \\ = \frac{d}{dt_1} \Big|_0 \cdots \frac{d}{dt_k} \Big|_0 f(x \exp(t_1 X_1) \cdots \exp(t_k X_k)g). \end{aligned}$$

Substitute  $\exp(sY)$  for  $g$  (where  $s \in \mathbb{R}$  and  $Y \in \mathfrak{g}$ ), and take the derivative with respect to  $s$  at  $s = 0$ ; we get

$$\begin{aligned} \frac{d}{ds} \Big|_0 \frac{d}{dt_1} \Big|_0 \cdots \frac{d}{dt_k} \Big|_0 f(x \exp(sY) \exp(t_1 X_1) \cdots \exp(t_k X_k)) \\ = \frac{d}{dt_1} \Big|_0 \cdots \frac{d}{dt_k} \Big|_0 \frac{d}{ds} \Big|_0 f(x \exp(t_1 X_1) \cdots \exp(t_k X_k) \exp(sY)). \end{aligned}$$

This implies

$$\tilde{Y}\tilde{X}_1 \cdots \tilde{X}_k f = \tilde{X}_1 \cdots \tilde{X}_k \tilde{Y}f,$$

which proves that  $[\tilde{Y}, D] = 0$  for all  $Y$  in  $\mathfrak{g}$ .

**2.2.14 THE CASIMIR ELEMENT.** What is the element in  $\mathcal{U}(\mathfrak{g})$  that corresponds to the Laplacian under the identification of  $\mathcal{U}(G)$  with  $D(G)$ ? We claim that it is the (*quadratic*) *Casimir element*  $\Omega$ , which is defined as follows. Let  $\text{End}(\mathfrak{g})$  denote the space of linear maps from  $\mathfrak{g}$  into itself. Let  $\mathfrak{g}^*$  be the dual space of  $\mathfrak{g}$ . Using the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ , we can construct the isomorphisms

$$\text{End}(\mathfrak{g}) \simeq \mathfrak{g} \otimes \mathfrak{g}^* \simeq \mathfrak{g} \otimes \mathfrak{g}.$$

Since  $\mathfrak{g} \otimes \mathfrak{g}$  is a subspace of the tensor algebra  $T(\mathfrak{g})$ , we have a  $G$ -equivariant map

$$\text{End}(\mathfrak{g}) \hookrightarrow T(\mathfrak{g}) \xrightarrow{/J(\mathfrak{g})} \mathcal{U}(\mathfrak{g}), \quad (2.2.15)$$

where  $/J(\mathfrak{g})$  denotes the quotient map with respect to the ideal  $J(\mathfrak{g})$  described in Section 2.2.10. The image of the identity map on  $\mathfrak{g}$  under the above composition is the Casimir element  $\Omega$ . Because the Casimir element originated from the identity map, it is in the center

$\mathcal{Z}(\mathfrak{g})$  of the universal enveloping algebra.<sup>4</sup>

The definition of  $\Omega$  does not depend on the particular basis for  $\mathfrak{g}$ . But it has a simple expression in terms of an orthonormal basis  $X_1, \dots, X_n$  for  $\mathfrak{g}$ . Let  $\theta_1, \dots, \theta_n$  be the dual basis for  $\mathfrak{g}^*$ . Then the identity map in  $\text{End}(\mathfrak{g})$  corresponds to  $\sum_{i=1}^n \theta_i \otimes X_i$  in  $\mathfrak{g}^* \otimes \mathfrak{g}$ . So the Casimir element is

$$\Omega = \sum_{i=1}^n X_i X_i. \quad (2.2.16)$$

So why is this the element that corresponds to the Laplacian under the identification of  $\mathcal{U}(\mathfrak{g})$  with  $D(G)$ ? Recall that the exponential map  $\exp : \mathfrak{g} \rightarrow G$  maps a neighborhood  $U$  of 0 in  $\mathfrak{g}$  diffeomorphically onto a neighborhood  $V$  of  $e$  in  $G$ . This provides us the *exponential chart* on  $V$ ; the coordinates  $(y_1, \dots, y_n)$  for  $g$  in  $V$  are the components of  $X = \exp^{-1}(g)$  with respect to the basis  $X_1, \dots, X_n$ ; in other words,  $g = \exp(\sum_{i=1}^n y_i X_i)$ . The exponential map we use here is the Lie-theoretic exponential map. There is also the Riemannian exponential map  $\text{Exp} : \mathfrak{g} \rightarrow G$  coming from differential geometry, which is also a local diffeomorphism near 0 in  $\mathfrak{g}$ . (For more on the Riemannian exponential map, see [54, Ch. 1, § 6]). The Riemannian exponential map depends on the metric, whereas the Lie-theoretic exponential map has nothing to do with the metric. But the two exponential maps agree if the metric is bi-invariant. This can be seen as follows. Let  $\nabla$  be the Riemannian connection so that  $\nabla_{\tilde{X}}(\tilde{Y}, \tilde{Z}) = \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \langle \tilde{Y}, \nabla_{\tilde{X}} \tilde{Z} \rangle$ . Using the identity (see [22, p. 2])

$$\begin{aligned} 2\langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle &= \tilde{X}\langle \tilde{Y}, \tilde{Z} \rangle + \tilde{Y}\langle \tilde{Z}, \tilde{X} \rangle - \tilde{Z}\langle \tilde{X}, \tilde{Y} \rangle \\ &\quad + \langle [\tilde{X}, \tilde{Y}], \tilde{Z} \rangle - \langle [\tilde{Y}, \tilde{Z}], \tilde{X} \rangle + \langle [\tilde{Z}, \tilde{X}], \tilde{Y} \rangle \end{aligned}$$

and the antisymmetry of the  $\text{ad}(\mathfrak{g})$ -action, one can check that

$$\nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2} [\tilde{X}, \tilde{Y}] \quad (2.2.17)$$

for all  $X$  and  $Y$  in  $\mathfrak{g}$ . In particular,  $\nabla_{\tilde{X}} \tilde{X} = 0$ . It follows that the geodesic  $\gamma_X(t)$ , such that  $\gamma_X(0) = e$  and  $\gamma'_X(0) = X$ , is a group homomorphism  $\mathbb{R} \rightarrow G$  (see [54, Ch. 2, Prop. 1.4]). By the uniqueness of one-parameter subgroups, we have  $\gamma_X(t) = \exp(tX)$ . This implies that the Riemannian exponential map is identical to the Lie-theoretic exponential map. Hence, the matrix  $[\eta_{ij}]$  of the metric under the exponential chart satisfies  $\eta_{ij}(e) = \delta_{ij}$  (Kronecker delta) and  $\partial_k \eta_{ij}(e) = 0$ . Therefore, the Laplacian at  $e$  takes the form

<sup>4</sup> The Lie group  $G$  acts on  $\mathfrak{g}$  by the adjoint action 2.1.2. This extends to the tensor algebra  $T(\mathfrak{g})$  by defining  $g \cdot (Y_1 \otimes \dots \otimes Y_k) = (g \cdot Y_1) \otimes \dots \otimes (g \cdot Y_k)$  for  $g$  in  $G$  and vectors  $Y_1, \dots, Y_k$  in  $\mathfrak{g}$ . This descends to a  $G$ -action on  $\mathcal{U}(\mathfrak{g})$ . The action of  $G$  on  $A \in \text{End}(\mathfrak{g})$  is given by  $(g \cdot A)(X) = g \cdot A(g^{-1} \cdot X)$ . The composition 2.2.15, under these actions, is  $G$ -equivariant. This owes mostly to the fact that the inner product  $\langle \cdot, \cdot \rangle$  is invariant. The identity map in  $\text{End}(\mathfrak{g})$  is obviously invariant under the  $G$ -action. So the Casimir element  $\Omega$  is a  $G$ -invariant element of  $\mathcal{U}(\mathfrak{g})$ ; this implies that  $\Omega$  is in the center of  $\mathcal{U}(\mathfrak{g})$ .



$\sum \partial_i^2 = \sum X_i X_i$ . Since the Laplacian is left-invariant, we have

$$\Delta_G = \sum \tilde{X}_i \tilde{X}_i$$

on  $G$ . This proves that

$$\tau(\Omega) = \Delta_G \quad (2.2.18)$$

where  $\tau$  is the algebra isomorphism 2.2.11.

**2.2.19 PETER-WEYL THEOREM.** Let  $C(G, \mathbb{C})$  denote the space of continuous  $\mathbb{C}$ -valued functions on  $G$ . A function  $f$  in  $C(G, \mathbb{C})$  is said to be *G-finite* if its orbit under the left-regular action of  $G$  spans a finite-dimensional subspace of  $C(G, \mathbb{C})$ . Denote the subspace of  $C(G, \mathbb{C})$  that consists of  $G$ -finite elements by  $C(G, \mathbb{C})_{\text{fin}}$ . The Peter-Weyl theorem [79] says that the space  $C(G, \mathbb{C})_{\text{fin}}$  of  $G$ -finite functions is dense in the space  $L^2(G, \mathbb{C})$  of square-integrable  $\mathbb{C}$ -valued functions on  $G$ . (The theorem is actually for any compact topological group  $G$ .)

All  $G$ -finite functions arise from finite-dimensional unitary representations over  $\mathbb{C}$ . Let us see why this is so. Suppose  $u : G \rightarrow \text{Aut}(V)$  is a finite-dimensional unitary representation over  $\mathbb{C}$ . A *representative function* of  $u$  is a function of the form

$$\begin{aligned} G &\rightarrow \mathbb{C}, \\ g &\mapsto \langle w, g^{-1} \cdot v \rangle_V, \end{aligned}$$

where  $w$  and  $v$  are vectors in  $V$  and  $\langle \cdot, \cdot \rangle_V$  is the inner product on a complex vector space  $V$ . Let  $\mathcal{M}_u$  be the space of representative functions of  $u$ . Then we have a linear isomorphism

$$\begin{aligned} \Psi_u : \mathcal{M}_u &\rightarrow V_u^* \otimes V_u, \\ f(g) = \langle v_1 | g^{-1} \cdot v_2 \rangle_V &\mapsto v_1^* \otimes v_2, \end{aligned} \quad (2.2.20)$$

where  $v_1^*$  is the linear functional defined by  $w \mapsto \langle v_1 | w \rangle$ . This is in fact an isomorphism of  $(G \times G)$ -vector spaces, where  $(g, h) \in G \times G$  acts on  $f \in \mathcal{M}_u$  by  $(g, h) \cdot f = L_g R_h f$ , and on  $v_1^* \otimes v_2 \in V_u^* \otimes V_u$  by  $(g, h) \cdot (v_1^* \otimes v_2) = \check{u}_g v_1^* \otimes u_h v_2$ . This shows that  $\mathcal{M}_u \subseteq C(G, \mathbb{C})_{\text{fin}}$ . But every  $G$ -finite function must arise in this way for the following reason. If  $f$  is a  $G$ -finite function, then  $f$  is an element of the finite-dimensional vector space  $V$  spanned by the orbit of  $f$  under the left-regular action of  $G$ . Let  $\phi$  be the linear functional on  $V$  given by evaluating the functions in  $V$  at the identity. Then

$$f(g) = \phi(g^{-1} \cdot f).$$

This shows that  $f$  is a representative function.

Note that the representative functions are smooth. Hence, the continuous functions that are  $G$ -finite are of class  $C^\infty$ . So let us write  $C(G, \mathbb{C})_{\text{fin}}$  as  $C^\infty(G, \mathbb{C})_{\text{fin}}$ .

To sum up, we have

$$C^\infty(G, \mathbb{C})_{\text{fin}} = \bigoplus_{u \in \widehat{G}} \mathcal{M}_u \simeq \bigoplus_{u \in \widehat{G}} V_u^* \otimes V_u, \quad (2.2.21)$$

where  $\hat{G}$  is the unitary dual of  $G$ . Then the Peter-Weyl theorem can be stated as

$$L^2(G, \mathbb{C}) \simeq \bigoplus_{u \in \hat{G}} V_u^* \otimes V_u, \quad (2.2.22)$$

where  $\bigoplus$  denotes the Hilbert space direct sum; the orthogonality of this direct sum is known as *Schur Orthogonality* (see [18, Thm. 2.3, p. 11]).

What follows from the isomorphism 2.2.22 is that the subspace  $L_{cl}^2(G, \mathbb{C})$  of  $L^2(G, \mathbb{C})$  that consists of the *class functions* (that is, functions that are invariant under the conjugation action of  $G$  on itself) in  $L^2(G, \mathbb{C})$  is spanned by the irreducible characters of  $G$ :

$$L_{cl}^2(G, \mathbb{C}) \simeq \bigoplus_{u \in \hat{G}} \mathbb{C} \cdot \chi_u. \quad (2.2.23)$$

2.2.24 Assume that the compact Lie group  $G$  is connected. Consider the following diagram:

$$\begin{array}{ccc} C^\infty(G, \mathbb{C})_{fin} & \xrightarrow{\Delta_G} & C^\infty(G, \mathbb{C})_{fin} \\ \parallel & & \parallel \\ C^\infty(G, \mathbb{C})_{fin} & \xrightarrow{\Omega} & C^\infty(G, \mathbb{C})_{fin} \\ (2.2.21) \parallel \{ & & \} \parallel (2.2.21) \\ \bigoplus_{u \in \hat{G}} V_u^* \otimes V_u & \xrightarrow{\Omega} & \bigoplus_{u \in \hat{G}} V_u^* \otimes V_u \end{array} \quad (2.2.25)$$

The top horizontal map is given by the canonical action of the Laplacian on smooth functions. (The representative functions The middle horizontal map is given by the Casimir element via the  $\mathcal{U}(\mathfrak{g})$ -action that is induced (through the algebra isomorphism 2.2.11) by the canonical  $D(G)$ -action on  $C^\infty(G, \mathbb{C})_{fin}$ ; as we have seen in Section 2.2.10, this  $\mathcal{U}(\mathfrak{g})$ -action on  $C^\infty(G, \mathbb{C})_{fin}$  coincides with that induced by the right-regular action of  $G$  on  $C^\infty(G, \mathbb{C})_{fin}$ . Now, under the vector space isomorphism 2.2.21, the action of  $G$  on the component  $V_u^* \otimes V_u$  that is compatible with the right-regular action of  $G$  on  $C^\infty(G, \mathbb{C})_{fin}$  is by  $1 \otimes u$ , where  $1$  denotes the trivial representation of  $G$  on  $V_u^*$ . Thus, the bottom horizontal map is given by the Casimir element via the  $\mathcal{U}(\mathfrak{g})$ -action obtained by extending the induced Lie algebra representation  $u_*$  on  $V_u$  to  $\mathcal{U}(\mathfrak{g})$ . With the above descriptions, the diagram 2.2.25 is commutative.

Owing to the connectedness of  $G$ , the induced Lie algebra representation  $u_*$  on  $V_u$  is irreducible.<sup>5</sup> Hence,  $V_u$  is irreducible under

<sup>5</sup> Suppose  $u_* : \mathfrak{g} \rightarrow \text{End}(V_u)$  is reducible. That means there is a nonzero proper subspace  $W$  of  $V_u$  that is invariant under  $u_*(\mathfrak{g})$ . Then, owing to Equation 2.2.6,  $W$  is invariant under  $u_g$  for  $g$  in the image of the exponential map. But this is sufficient to conclude that  $W$  is invariant under  $u_g$  for all  $g$  in  $G$  because the image of the exponential map generates the connected Lie group  $G$ ; see [92, Thm. 3.18, p.93].

the  $\mathcal{U}(\mathfrak{g})$ -action induced by  $u_*$ . So, by Schur's lemma, the Casimir element, which is in the center of  $\mathcal{U}(\mathfrak{g})$ , acts by a scalar on the summand  $V_u^* \otimes V_u$ ; call that scalar  $\Omega(u)$ . These scalars are the eigenvalues of  $\Delta_G$  on  $C^\infty(G, \mathbb{C})_{\text{fin}}$ . The Peter-Weyl theorem 2.2.22 then implies that there is an orthonormal basis for  $L^2(G, \mathbb{C})$  consisting of smooth eigenfunctions of  $\Delta_G$ , and that the multiplicity of the  $\Omega(u)$ -eigenfunction is equal to  $\dim(u)^2$ . Hence, the heat-trace of  $\Delta_G$  is given by

$$\text{tr}(e^{t\Delta_G}) = \text{tr}(e^{t\Omega}) = \sum_{u \in \widehat{G}} \dim(u)^2 e^{t\Omega(u)}. \quad (2.2.26)$$

In view of the isomorphism 2.2.23, we can obtain the spectrum  $\{\Omega(u)\}_{u \in \widehat{G}}$  by applying the Laplacian to the irreducible characters  $\chi_u$ . Moreover, since  $\chi_u(e) = \dim(u)$ , having a complete information on the irreducible characters allows us to calculate the heat trace 2.2.26. This becomes a feasible task with H. Weyl's formula for the irreducible characters, which brings us to the last topic of our review.

### 2.3 THE WEYL CHARACTER FORMULA AND THE SPECTRUM OF THE LAPLACIAN

2.3.1 Throughout this section we assume that the compact Lie group  $G$  is connected. We continue to denote the selected bi-invariant metric on  $G$  by  $\langle \cdot, \cdot \rangle$ .

2.3.2 H. Weyl's approach [96–98] to characterizing the irreducible representations of  $G$  starts by considering a connected maximal commutative subgroup  $T$  of  $G$ . Such a subgroup is known as a *maximal torus* of  $G$  because a compact connected commutative Lie group is isomorphic to a torus (see [18, Prop. 15.3, p. 87]); hence,

$$T \simeq \prod_{i=1}^{\dim T} U(1).$$

It is a fact that the conjugates of  $T$  exhaust all maximal tori of  $G$  and that their union  $\bigcup_{g \in G} g^{-1}Tg$  is equal to  $G$  (see [18, Thm. 16.4, p. 103]). Thus, the dimension of a maximal torus depends only on  $G$ ; it is called the *rank* of  $G$ .

The elements of  $G$  that preserve the selected maximal torus  $T$  under conjugation constitute the normalizer  $N_G(T)$  of  $T$  relative to  $G$ . The elements of  $N_G(T)$  that act trivially on  $T$  under conjugation constitutes  $T$ . The quotient

$$W := N_G(T)/T$$

is called the *Weyl group* of  $G$ .

If  $f$  is a smooth function on  $T$ , then an element  $gT$  of the Weyl group acts on  $f$  by  $(gT \cdot f)(t) = f(gtg^{-1})$  where  $x$  in  $T$ . A function

on  $T$  can be extended to a class function on  $G$  if and only if it is invariant under the  $W$ -action. Thus, the restriction to  $T$  gives a linear isomorphism

$$L^2_{\text{cl}}(G, \mathbb{C}; \mu_G) \xrightarrow{\sim} L^2(T, \mathbb{C}; \mu_T)^W,$$

where  $\mu_G$  and  $\mu_T$  are the normalized Haar measures on  $G$  and  $T$ , respectively, and the decoration  $W$  is for the subspace of  $W$ -invariants. H. Weyl found that there is some smooth function  $\mathcal{D}$  on  $T$  such that if we modify the measure on  $T$  by the factor  $\mathcal{D}/|W|$  then the restriction map,

$$L^2_{\text{cl}}(G, \mathbb{C}; \mu_G) \xrightarrow{\sim} L^2(T, \mathbb{C}; \frac{\mathcal{D}}{|W|} \mu_T)^W, \quad (2.3.3)$$

is unitary. By carefully studying the image of an irreducible character  $\chi$  of  $G$  under this map, H. Weyl obtained a formula for  $\chi$  in terms of the irreducible characters of  $T$ . (We will look into this in more detail in Section 2.3.24.) Bearing in mind that the set of irreducible characters of  $T$  can be identified with the unitary dual  $\hat{T}$  of  $T$ , a corollary of the character formula is that there is a one-to-one correspondence between  $\hat{G}$  and  $\hat{T}/W$ . The *highest weight theory*, which we shall review shortly, gives a means to parametrize  $\hat{T}/W$ .

As we review some key notions surrounding the character formula, we shall closely follow the global and analytic approach of H. Weyl. There is a corresponding infinitesimal and algebraic approach set forth by E. Cartan [19]; for that, we refer to [88].

**2.3.4 WEIGHTS.** Recall that any representation of a compact connected Lie group  $G$  on a Hilbert space is a direct sum of finite-dimensional irreducible representations. If  $G$  is *commutative*, so that  $G$  is isomorphic to a torus

$$T = \prod_{i=1}^r U(1),$$

then the irreducible representations are all 1-dimensional<sup>6</sup>. Hence, for tori, the following notions are identical: (a) a 1-dimensional representation, (b) an irreducible representation, (c) the character of an irreducible representation. Thus, the unitary dual  $\hat{T}$  of  $T$  has a natural structure of a commutative group.

An irreducible character of an  $r$ -dimensional torus must be of the form

$$\begin{aligned} \theta : T = \prod_{i=1}^r U(1) &\rightarrow \mathbb{C}, \\ x = (e^{ix_1}, \dots, e^{ix_r}) &\mapsto \theta(x) = e^{i(n_1 x_1 + \dots + n_r x_r)}, \end{aligned} \quad (2.3.5)$$

where  $(n_1, \dots, n_r) \in \mathbb{Z}^r$ . As we have noted earlier, the character  $\theta$

<sup>6</sup> Suppose  $u : T \rightarrow \text{Aut}(V)$  is an irreducible representation. Since  $T$  is commutative,  $u(x) : V \rightarrow V$  is an intertwiner for any  $x$  in  $T$ . Thus, by Schur's lemma, each  $u(x)$  is a scalar multiple of the identity map on  $V$ . So any 1-dimensional subspace of  $V$  is an invariant subspace. Since  $V$  is irreducible,  $V$  must be 1-dimensional.

is just the irreducible representation of  $T$  on  $\mathbb{C}$  where  $x \in T$  acts on  $z \in \mathbb{C}$  by  $x \cdot z = \theta(x)z$ . The induced Lie algebra representation of  $H \in \mathfrak{t}$  is given by:

$$H \cdot z = \theta_*(H)z = \left. \frac{d}{ds} \right|_0 \theta(\exp sH)z. \quad (2.3.6)$$

If we identify the Lie algebra  $\mathfrak{t}$  of  $T$  with  $\mathbb{R}^r$ , then the exponential map takes the form

$$\begin{aligned} \exp : \quad \mathfrak{t} \simeq \mathbb{R}^r &\rightarrow T = \prod_{i=1}^r U(1), \\ (x_1, \dots, x_r) &\mapsto (e^{ix_1}, \dots, e^{ix_r}). \end{aligned} \quad (2.3.7)$$

Using the maps and equations 2.3.5–2.3.7, we have, for  $H$  in  $\mathfrak{t}$ ,

$$\theta_*(H) = i(n_1 x_1 + \dots + n_r x_r), \quad (2.3.8)$$

where  $x_i$  is the  $i$ th component of the vector in  $\mathbb{R}^r$  that corresponds to  $H$  under the identification of  $\mathfrak{t}$  with  $\mathbb{R}^r$ . We see that  $\theta_*$  is a linear function  $\mathfrak{t} \rightarrow \mathbb{C}$  whose value is purely imaginary; it is called the *complex weight* of  $\theta$ . The corresponding *real weight* is the linear functional

$$\mu := -i\theta_*.$$

In terms of  $\mu$ , the action of  $H \in \mathfrak{t}$  on  $z \in \mathbb{C}$  is given by

$$H \cdot z = i\mu(H)z.$$

We will be dealing with real weights most of the time, so when we simply say “weights” it is to be understood as real weights.

We denote by  $\Lambda_T \subseteq \mathfrak{t}^*$  the set of weights that come from irreducible characters of  $T$ ; in other words,  $\Lambda_T$  is the image of the map

$$\begin{aligned} \hat{T} &\rightarrow \mathfrak{t}^*, \\ \theta &\mapsto -i\theta_*. \end{aligned}$$

We can conclude from Equation 2.3.8 that

$$\Lambda_T = \{ \mu \in \mathfrak{t}^* \mid \mu(H) \in 2\pi\mathbb{Z} \text{ for all } H \in \mathfrak{t} \cap \exp^{-1}\{e\} \}.$$

This is a lattice of rank  $r := \dim(T)$ . If we let

$$\Lambda_e := \{ H \in \mathfrak{t} \mid \exp(2\pi H) = e \},$$

then  $\Lambda_T$  is the lattice that is dual to  $\Lambda_e$  in the following sense:

$$\Lambda_T = \{ \mu \in \mathfrak{t}^* \mid \mu(H) \in \mathbb{Z} \text{ for all } H \in \Lambda_e \}.$$

Moving on to the general case, allow the compact connected Lie group  $G$  to be noncommutative. Let  $u : G \rightarrow \text{Aut}(V)$  be a finite-dimensional complex representation. Upon restricting  $u$  to a maximal torus  $T$  in  $G$ , the representation space  $V$  is decomposed into 1-dimensional invariant subspaces, each yielding a weight of  $T$ . Grouping the 1-dimensional invariant subspaces according to their weights,

we can write  $V$  as a direct sum,

$$V = V_{\mu_1} \oplus \cdots \oplus V_{\mu_\ell}, \quad (2.3.9)$$

where  $\mu_k$  ( $1 \leq k \leq \ell$ ) are distinct weights. Put in another way, the direct sum 2.3.9 is the eigenspace decomposition under the  $\mathfrak{t}$ -action induced by  $\mathfrak{u}$ ; an element  $H \in \mathfrak{t}$  acts on  $v \in V_{\mu_k}$  as follows

$$H \cdot v = i\mu_k(H)v.$$

The linear functionals  $\mu_1, \dots, \mu_\ell$  are called the *weights* of  $\mathfrak{u}$ , and the subspaces  $V_{\mu_k}$  are called the *weight spaces* of  $V$ . Vectors in  $V_{\mu_k}$  are called *weight vectors* with weight  $\mu_k$ . The decomposition 2.3.9 is called the *weight space decomposition* of  $V$ .

2.3.10 We may talk about the weights of a representation  $\mathfrak{u}$  even if it is a real representation, by declaring its weights to be those of the complexification of  $\mathfrak{u}$ , that is, the representation on  $V_{\mathbb{C}} = V \otimes \mathbb{C}$  obtained by extending the action of  $\mathfrak{u}_g$  ( $g \in G$ ) by  $\mathbb{C}$ -linearity.

2.3.11 ROOTS. The most important example of weights coming from a real representation are the weights of the adjoint representation  $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ . To find the weights, we must extend the representation space to the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g}$  and restrict the domain of the representation to a maximal torus  $T$  of  $G$ . That gives us the weight space decomposition:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C},0} \oplus \mathfrak{g}_{\mathbb{C},\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\mathbb{C},\alpha_k}, \quad (2.3.12)$$

where  $\mathfrak{g}_{\mathbb{C},\alpha}$  denotes the weight space corresponding to the weight  $\alpha$ ; the nonzero weights  $\alpha_1, \dots, \alpha_k$  of the adjoint representation are called the *roots* of  $G$ ; the space  $\mathfrak{g}_{\mathbb{C},\alpha_i}$  is called the *root space* associated to the root  $\alpha_i$  of  $G$ , and the decomposition 2.3.12 is called the *root space decomposition* of  $\mathfrak{g}_{\mathbb{C}}$ . We shall denote the set of roots of  $G$  by  $\Phi$ .

Note that the zero weight space  $\mathfrak{g}_{\mathbb{C},0}$  is the centralizer of  $\mathfrak{t}_{\mathbb{C}}$ , that is, the subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  that commutes with every element in  $\mathfrak{t}_{\mathbb{C}}$ . Because  $\mathfrak{t}$  is a maximal commutative subalgebra of  $\mathfrak{g}$  (otherwise,  $T$  cannot be a maximal and commutative subgroup of  $G$ ), we have

$$\mathfrak{g}_{\mathbb{C},0} = \mathfrak{t}_{\mathbb{C}}.$$

Therefore, the root space decomposition 2.3.12 can be rewritten as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C},\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\mathbb{C},\alpha_k}. \quad (2.3.13)$$

Using the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ , one can find, for each linear functional  $\mu \in \mathfrak{t}^*$ , a unique vector  $X_{\mu} \in \mathfrak{t}$  such that

$$\mu(H) = \langle X_{\mu}, H \rangle, \quad \forall H \in \mathfrak{t}. \quad (2.3.14)$$

This gives a one-to-one correspondence between  $\mathfrak{t}$  and  $\mathfrak{t}^*$ ; and the

inner product  $\langle \cdot, \cdot \rangle$  can be transferred to  $\mathfrak{t}^*$  by setting

$$\langle \mu, \nu \rangle := \langle X_\mu, X_\nu \rangle.$$

This inner product on  $\mathfrak{t}^*$  is also invariant under the  $\check{\text{Ad}}(G)$ -action (see Section 2.2.5 for the notation). We adopt the notation

$$X_\mu(\nu) := \nu(X_\mu).$$

The set  $\Phi$  of roots of  $G$  satisfies the following properties (see [18, Ch. 19 and 23]):

- (i) Each root space  $\mathfrak{g}_{\mathbb{C}, \alpha}$  is 1-dimensional.
- (ii) If  $\alpha \in \Phi$ , then  $\lambda\alpha \in \Phi$  if and only if  $\lambda = \pm 1$ .
- (iii) The root spaces  $\mathfrak{g}_{\mathbb{C}, \alpha}$  and  $\mathfrak{g}_{\mathbb{C}, \beta}$  are orthogonal if  $\alpha \neq \pm\beta$ .
- (iv) The subspace  $[\mathfrak{g}_{\mathbb{C}, \alpha}, \mathfrak{g}_{\mathbb{C}, -\alpha}]$  is a 1-dimensional subspace of  $\mathfrak{t}_{\mathbb{C}}$ . And the subspace  $\mathfrak{g}_{\mathbb{C}, \alpha} \oplus \mathfrak{g}_{\mathbb{C}, -\alpha} \oplus [\mathfrak{g}_{\mathbb{C}, \alpha}, \mathfrak{g}_{\mathbb{C}, -\alpha}]$  forms a Lie algebra isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ .
- (v) Let  $H_\alpha$  be the unique element in  $[\mathfrak{g}_{\mathbb{C}, \alpha}, \mathfrak{g}_{\mathbb{C}, -\alpha}]$  such that

$$\alpha(H_\alpha) = 2.$$

Then  $H_\alpha$  is a vector in  $\mathfrak{t}$  that satisfies  $\exp(2\pi H_\alpha) = e$ . And  $\beta(H_\alpha) \in \mathbb{Z}$  for all  $\beta$  in  $\Phi$ .

- (vi) For  $\alpha$  in  $\Phi$ , the linear transformation  $s_\alpha : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ ,  $\mu \mapsto \mu - \mu(H_\alpha)\alpha$ , is a reflection that sends  $\alpha$  to  $-\alpha$ . Moreover,  $s_\alpha$  permutes the roots.
- (vii) The compact connected Lie group  $G$  is semisimple if and only if  $\Phi$  spans  $\mathfrak{t}^*$ .

The vector  $H_\alpha$  is called the *coroot* of  $\alpha$ . The reflections  $s_\alpha$  are called the *Weyl reflections*; they are precisely the isometries of  $\mathfrak{t}^*$  that are induced by the conjugation action of the Weyl group  $W = N_G(T)/T$  on  $T$ .

According to property (ii) above, the roots come in pairs  $(\alpha, -\alpha)$ . It is helpful to distinguish one from each pair and call the selected ones “positive”. To that end, divide  $\mathfrak{t}^*$  into disconnected regions by excluding from  $\mathfrak{t}^*$  the hyperplanes  $h_\alpha := \ker(1 - s_\alpha)$ ,  $\alpha \in \Phi$ . So we have the set  $\mathfrak{t}^* \setminus \bigcup_{\alpha \in \Phi} h_\alpha$ , and the Weyl group acts freely on it. A connected component of  $\mathfrak{t}^* \setminus \bigcup_{\alpha \in \Phi} h_\alpha$  is called a *Weyl chamber*. Pick a Weyl chamber  $K^\circ$ . It is mapped to another Weyl chamber under the  $W$ -action; in fact,  $K^\circ$  is a fundamental domain of the free  $W$ -action on  $\mathfrak{t}^* \setminus \bigcup_{\alpha \in \Phi} h_\alpha$ . Once we have our choice for  $K^\circ$ , we define the set of *positive roots* as

$$\Phi^+ := \{ \alpha \in \Phi : \langle \alpha, \nu \rangle > 0, \forall \nu \in K^\circ \}.$$

Note that  $\alpha \in \Phi^+$  implies  $-\alpha \in \Phi \setminus \Phi^+$ . So we call the roots in  $\Phi^- := \Phi \setminus \Phi^+$  the *negative roots*. We have a disjoint union:

$$\Phi = \Phi^+ \sqcup \Phi^-.$$

This decomposition of  $\Phi$  into positive and negative roots depends on which Weyl chamber we select for  $K^\circ$ . From now on, we shall assume that the choice for  $K^\circ$  has been made. The closure  $K$  of  $K^\circ$  is called the *fundamental Weyl chamber* of our choice.

*Remark.* Since roots are special type of weights, they span a sublattice  $\Lambda_\Phi$  of  $\Lambda_T$ , called the *root lattice* of  $G$ . The lattice  $\Lambda_{\text{coroot}}$  spanned by the coroots is called the *coroot lattice* of  $G$ ; it is a sublattice of  $\Lambda_e$ .

**2.3.15 FUNDAMENTAL WEIGHTS.** The coroots, or rather, their dual vectors relative to the inner product can be used to parametrize the lattice  $\Lambda_T$ . To that end, choose a special basis for the lattice  $\Lambda_\Phi$  spanned by the roots. We do this by collecting the positive roots that cannot be written as a sum of other positive roots; such roots are said to be *simple*. The set  $\Phi_s^+$  of simple roots is a  $\mathbb{Z}$ -basis for the root lattice  $\Lambda_\Phi$ , and an  $\mathbb{R}$ -basis for  $\Lambda_\Phi \otimes \mathbb{R}$ . Suppose

$$\Phi_s^+ = \{\alpha_1, \dots, \alpha_\ell\}.$$

Consider the set of corresponding simple coroots in  $\mathfrak{t}$ :

$$\Sigma = \{H_{\alpha_1}, \dots, H_{\alpha_\ell}\}.$$

Let  $\Sigma^* = \{\lambda_1, \dots, \lambda_\ell\}$  be the dual basis for  $\Lambda_\Phi \otimes \mathbb{R}$  relative to  $\Sigma$ , so that

$$\lambda_i(H_{\alpha_j}) = \delta_{ij}.$$

The linear functionals in  $\Sigma^*$  are called the *fundamental (dominant) weights* of  $G$ . They satisfy the following properties (see [18, Ch. 21, Ch. 23; 34, § 3.11]):

- (i) They lie in the boundary walls of the fundamental Weyl chamber  $K$ .
- (ii) An element  $\mu = \sum_{i=1}^{\ell} x_i \lambda_i$  in  $\mathfrak{t}^*$  lies in  $K$  if and only if  $x_i \geq 0$  for all  $i$ .
- (iii) The sum of the fundamental weights is equal to half the sum of the positive roots:

$$\rho := \lambda_1 + \dots + \lambda_\ell = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

- (iv) The fundamental weights are part of the set

$$\Lambda_{\mathfrak{g}} = \{ \mu \in \mathfrak{t}^* \mid \mu(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi \}.$$

Note that  $\Lambda_{\mathfrak{g}}$  is dual to  $\Lambda_{\text{coroot}}$  in the sense that

$$\Lambda_{\mathfrak{g}} = \{ \mu \in \mathfrak{t}^* \mid \mu(H) \in \mathbb{Z} \text{ for all } H \in \Lambda_{\text{coroot}} \}.$$



We have

$$\Lambda_T \subseteq \Lambda_{\mathfrak{g}}.$$

By duality, we have

$$\Lambda_e \supseteq \Lambda_{\text{coroot}}.$$

- (v) If the compact connected Lie group  $G$  is semisimple, then the quotient group  $\Lambda_e/\Lambda_{\text{coroot}}$  is isomorphic to the fundamental group of  $G$ :

$$\Lambda_e/\Lambda_{\text{coroot}} \simeq \pi_1(G).$$

By duality,  $\Lambda_{\mathfrak{g}}/\Lambda_T$  is isomorphic to the unitary dual of  $\pi_1(G)$ :

$$\Lambda_{\mathfrak{g}}/\Lambda_T \simeq \widehat{\pi_1(G)}.$$

We distinguish  $\Lambda_{\mathfrak{g}}$  and  $\Lambda_T$  by calling their elements the *algebraically integral weights* and the *analytically integral weights* of  $G$ , respectively. The weights in  $\Lambda_{\mathfrak{g}} \cap K$  are said to be *dominant*; the ones in  $\Lambda_{\mathfrak{g}} \cap K^\circ$  are said to be *strictly dominant*. Note that

$$\Lambda_{\mathfrak{g}} \cap K^\circ = \rho + (\Lambda_{\mathfrak{g}} \cap K). \quad (2.3.16)$$

**2.3.17 PARTIAL ORDERING ON THE WEIGHT LATTICE.** We have put some effort in reviewing the notions surrounding the roots of  $G$ . This is because they can be used to give a partial ordering on the weight lattice  $\Lambda_T$  and, thus, allow us to talk about “highest weights” in a finite-dimensional representation of  $G$ . To wit, for  $\mu$  and  $\lambda$  in  $\Lambda_T$ , we say that  $\lambda$  is *higher* than  $\mu$  and write  $\mu \lesssim \lambda$  if and only if  $\lambda$  can be written as

$$\lambda = \mu + \sum_{\alpha \in \Phi^+} n_\alpha \alpha$$

for some nonnegative integers  $n_\alpha$ . That this defines a transitive relation on  $\Lambda_T$  is immediate. That this is a reflexive relation is equivalent to saying that if  $\sum_{\alpha \in \Phi^+} c_\alpha \alpha = 0$  with  $c_\alpha \geq 0$  for all  $\alpha \in \Phi^+$  then  $c_\alpha = 0$  for all  $\alpha \in \Phi^+$ ; for a proof that this is the case, see [34, p. 147]. In short, we do have a partial ordering on  $\Lambda_T$ .

Now suppose we have a finite-dimensional representation  $u$  of  $G$ . Let  $A = \{\mu_1, \dots, \mu_k\}$  be the set of weights of this representation. We say that  $\mu_i \in A$  is a *highest weight* of this representation if it is a maximal element of  $A$  with respect to the partial ordering  $\lesssim$ .

**2.3.18 WEYL INTEGRATION FORMULA.** Since the characters of  $G$  are class functions, their restrictions to  $T$  are  $W$ -invariant; hence, we have a linear map

$$L_{\text{cl}}^2(G, \mathbb{C}; \mu_G) \rightarrow L^2(T, \mathbb{C}; \mu_T)^W,$$

where  $\mu_G$  and  $\mu_T$  are any Haar measures on  $G$  and  $T$ , respectively. A  $W$ -invariant function on  $T$  extends to a class function on  $G$ , so the

above map is a linear isomorphism. H. Weyl found out that a slight modification of the measure on  $T$  gives a unitary map

$$L_{\text{cl}}^2(G, \mathbb{C}; \mu_G) \xrightarrow{\sim} L^2(T, \mathbb{C}; c\mathcal{D}\mu_T)^W,$$

where  $c$  is the constant  $\mu_G(G)/(\mu_T(T)|W|)$ , and  $\mathcal{D}$  is the function on  $T$  defined by

$$\mathcal{D}(x) = \prod_{\alpha \in \Phi^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})$$

where  $H \in \mathfrak{t} \cap \exp^{-1}\{x\}$ . This is a consequence of the celebrated *Weyl Integration Formula*:

$$\int_G f(g) \mu_G(g) = \frac{1}{|W|} \frac{\mu_G(G)}{\mu_T(T)} \int_T \left[ \int_{G/T} f(gxg^{-1}) \mu_{G/T}(\bar{g}) \right] |\mathcal{D}(x)|^2 \mu_T(x), \quad (2.3.19)$$

which holds for any integrable function  $f$  on  $G$ ; here  $\bar{g}$  denotes the image of  $g$  under the quotient map  $\pi : G \rightarrow G/T$  and  $\mu_{G/T}$  is the *quotient measure* on  $G/T$  defined by  $\mu_{G/T}(U) = \mu_G(\pi^{-1}(U))$  for any open set  $U$  in  $G/T$ . For a proof of the integration formula, see [34, Thm. 3.14.1, p. 185]. If  $f$  is a class function, then the integration formula simplifies to:

$$\int_G f(g) \mu_G(g) = \frac{1}{|W|} \frac{\mu_G(G)}{\mu_T(T)} \int_T f(t) |\mathcal{D}(t)|^2 \mu_T(t). \quad (2.3.20)$$

There is a corresponding formula for integration over  $\mathfrak{g}$ . Let  $\omega_G$  and  $\omega_T$  be the invariant volume forms on  $G$  and  $T$ , respectively, such that  $\int_G \omega_G = \mu_G(G)$  and  $\int_T \omega_T = \mu_T(T)$ . The values of  $\omega_G$  and  $\omega_T$  at the identity determines a volume form for  $\mathfrak{g}$  and  $\mathfrak{t}$ , respectively. Denote the Lebesgue measure they define on  $\mathfrak{g}$  and  $\mathfrak{t}$  by  $dX$  and  $dH$ , respectively. Then, for any integrable function  $\phi$  on  $\mathfrak{g}$ ,

$$\int_{\mathfrak{g}} \phi(X) dX = \frac{1}{|W|} \frac{\mu_G(G)}{\mu_T(T)} \int_{\mathfrak{t}} \left( \int_{G/T} \phi(\text{Ad}_g H) \mu_{G/T}(\bar{g}) \right) |\delta(H)|^2 dH, \quad (2.3.21)$$

where

$$\delta(H) = \prod_{\alpha \in \Phi^+} \alpha(H). \quad (2.3.22)$$

See [34, Thm. 3.14.1, p. 185] for a proof. If  $\phi$  is  $\text{Ad}(G)$ -invariant, then

$$\int_{\mathfrak{g}} \phi(X) dX = \frac{1}{|W|} \int_{\mathfrak{t}} \phi(H) |\delta(H)|^2 dH. \quad (2.3.23)$$

**2.3.24 WEYL CHARACTER FORMULA.** Notice that the function  $\mathcal{D}$  is antisymmetric with respect to the  $W$ -action. Meanwhile, the restriction of a class function  $f$  on  $G$  to  $T$  is  $W$ -invariant. So the product of  $\mathcal{D}$  and the restriction  $f|_T$  gives a  $W$ -antisymmetric function. This gives us a vector space isomorphism

$$\begin{array}{ccc} L_{\text{cl}}^2(G, \mathbb{C}; \mu_G) & \xrightarrow{\sim} & L^2(T, \mathbb{C}; \frac{1}{|W|} \mu_T)^{-W}, \\ f & \mapsto & f|_T \mathcal{D}, \end{array} \quad (2.3.25)$$

where  $L^2(T, \mathbb{C}; \frac{1}{|W|}\mu_T)^{-W}$  denotes the subspace of  $L^2(T, \mathbb{C}; \frac{1}{|W|}\mu_T)$  that consists of the  $W$ -antisymmetric elements. If we assume that  $\mu_G$  and  $\mu_T$  are the normalized Haar measures, then the integral formula 2.3.20 implies that the map 2.3.25 is unitary. Now, owing to the isomorphism 2.2.23, the set

$$\{\chi_u \mid u \in \widehat{G}\}$$

of irreducible characters of  $G$  is an orthogonal basis for  $L^2_{cl}(G, \mathbb{C}; \mu_G)$ . It is, in fact, an orthonormal basis (see [18, Thm. 2.4, p. 12]). This basis yields, through the unitary map 2.3.25, an orthonormal basis for  $L^2(T, \mathbb{C}; \frac{1}{|W|}\mu_T)^{-W}$ . H. Weyl found that the orthonormal basis for  $L^2(T, \mathbb{C}; \frac{1}{|W|}\omega_T)^{-W}$  thus obtained is

$$\left\{ \sum_{w \in W} \det(w) \theta_{w \cdot \lambda} \mid \lambda \in \Lambda_T \cap K^\circ \right\},$$

where  $\theta_\lambda$  denotes the irreducible character of  $T$  defined by  $\theta_\lambda(x) = e^{i\lambda(H)}$  ( $H \in \mathfrak{t} \exp^{-1}\{x\}$ ). Note that the  $W$ -orbits of weights in  $\Lambda_T \cap K^\circ$  are free and that these orbits exhaust all free orbits in  $\Lambda_T$ . Hence, for each character  $\chi_u$  of  $u$  in  $\widehat{G}$ , there is a unique weight  $\lambda$  in  $\Lambda_T \cap K^\circ$  such that

$$\chi_u \upharpoonright_T \mathcal{D} = \sum_{w \in W} \det(w) \theta_{w \cdot \lambda}, \quad (2.3.26)$$

and vice versa. More precisely, H. Weyl found that  $u$  and  $\lambda$  in the above equation are related by:

$$\lambda = \mu + \rho, \quad (2.3.27)$$

where  $\mu$  is a highest weight of  $u$  and  $\rho$  is the sum of the fundamental dominant weights. The uniqueness of  $\lambda$  determined by  $u$  implies the uniqueness of the highest weight  $\mu$ ; moreover, by Equation 2.3.16, the highest weight  $\mu$  lies in  $\Lambda_T \cap K$ . In short, we have a one-to-one correspondence:

$$\begin{aligned} \widehat{G} &\leftrightarrow \Lambda_T \cap K, \\ u &\mapsto \text{highest weight } \mu \text{ of } u. \end{aligned} \quad (2.3.28)$$

For this reason, the representation space of an irreducible representation of  $G$  that has highest weight  $\mu$  is often called a *highest weight module* with highest weight  $\mu$ .

We point out again that the one-to-one correspondence 2.3.28 is a consequence of Equations 2.3.26 and 2.3.27 which are collectively referred to as the *Weyl Character Formula*. They are usually combined and presented as

$$\chi_u \upharpoonright_T = \frac{\text{Alt}_{\mu+\rho}}{\mathcal{D}} \quad (2.3.29)$$

where

$$\text{Alt}_\lambda := \sum_{w \in W} \det(w) \theta_{w \cdot \lambda}.$$

Applying the Weyl Character Formula to the trivial representation, whose highest weight and character are  $\mu = 0$  and  $\chi_\mu \equiv 1$ , we get

$$\mathcal{D} = \text{Alt}_\rho. \quad (2.3.30)$$

This is known as the *Weyl Denominator Formula*.

Inserting the expression 2.3.30 for  $\mathcal{D}$  back into the character formula 2.3.29 and taking the limit  $\lim_{x \rightarrow e} \chi_\mu \upharpoonright_T(x)$  yields the *Weyl Dimension Formula* for the dimension of the highest weight module  $V(\mu)$  with highest weight  $\mu$ :

$$\dim V(\mu) = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \mu + \rho \rangle}{\langle \alpha, \rho \rangle}. \quad (2.3.31)$$

*Remark.* The significance of the weight  $\rho$  is that it is the highest weight of the representation of  $T$  on the spinor space  $\mathbb{S}$  of  $\text{Cl}(\mathfrak{p})$ , where  $\mathfrak{p}$  is the orthogonal complement of  $\mathfrak{t}$  in  $\mathfrak{g}$ ; the action of  $T$  on  $\mathbb{S}$  is by the group homomorphism  $T \rightarrow \text{Spin}(\mathfrak{p})$  that is the lift of the homomorphism  $T \rightarrow \text{SO}(\mathfrak{p})$  given by the adjoint action of  $T$  on  $\mathfrak{p}$ . (We discuss this representation in detail on pages 126–127.)

$$\begin{array}{ccc} & & \text{Spin}(\mathfrak{p}) \\ & \nearrow & \downarrow \\ T & \xrightarrow{\text{Ad}} & \text{SO}(\mathfrak{p}) \end{array}$$

The ubiquitous appearance of the weight  $\rho$  in the representation theory of  $G$  may be taken as a suggestion for using Dirac operators instead of the Casimir element in characterizing the irreducible representations of  $G$ .

2.3.32 It is worthwhile to give R. Bott's description [16] of the Weyl Character Formula using the representation ring. Let  $[V]$  denote the equivalence class of finite-dimensional  $G$ -vector spaces that are  $G$ -isomorphic to  $V$ . Let  $R(G)^+$  denote the set of such equivalence classes modulo the relation

$$\begin{aligned} [V] + [W] &= [V \oplus W], \\ [V][W] &= [V \otimes W]. \end{aligned} \quad (2.3.33)$$

Then *representation ring*  $R(G)$  of  $G$  is the abelian group generated by  $R(G)^+$ . As a  $\mathbb{Z}$ -module, we have

$$R(G) = \bigoplus_{u \in \widehat{G}} \mathbb{Z} \cdot [V_u],$$

where  $V_u$  denotes the representation space of  $u$ . If we identify  $[V]$  with the character  $\chi_V$  of the representation on  $V$ , then the relations in 2.3.33 are true equations of characters. For this reason,  $R(G)$  is also known as the *character ring*; in this point of view, the elements of  $R(G)$  are called *virtual characters*.

The character ring is functorial; if  $\iota : K \rightarrow G$  is a homomorphism of compact Lie groups, then the composition of a representation of  $G$  with  $\iota$  induces a ring homomorphism

$$\iota^* : R(G) \rightarrow R(K).$$

If  $\iota$  is the inclusion map of a Lie subgroup, then  $\iota^*$  is just the restriction map.

Now consider the representation ring of a maximal torus  $T$  of  $G$ . The action of the Weyl group  $W$  on  $T$  induces a  $W$ -action on  $R(T)$ . The isomorphism 2.3.25 can then be stated as follows [16, Thm. A, p. 175]: Let  $\iota : T \hookrightarrow G$  be the inclusion map of a maximal torus  $T$  of  $G$ . The induced homomorphism  $\iota^* : R(G) \rightarrow R(T)$  yields a ring isomorphism

$$R(G) \xrightarrow[\iota^*]{\sim} R(T)^W,$$

where  $R(T)^W$  denotes the ring of  $W$ -invariants in  $R(T)$ .

An element  $\chi$  of  $R(T)$  is said to be *W-alternating* if  $w \cdot \chi = \text{sgn}(w)\chi$ . Denote by  $R(T)^{-W}$  the subspace of  $R(T)$  that consists of the  $W$ -alternating elements. Of particular interest is the  $W$ -alternating element

$$[\Lambda] := [\mathbb{C}_\rho] \prod_{\alpha \in \Phi^+} (1 - [\mathbb{C}_{-\alpha}]),$$

where  $[\mathbb{C}_\mu]$  denotes the irreducible  $T$ -vector space with weight  $\mu$ . The Weyl Character Formula can then be restated as follows [16, Thm. D, p. 178, Cor. 6.1, p. 178]: Suppose  $\pi_1(G)$  has no 2-torsion.

- (a) The alternating element  $[\Lambda]$  generates  $R(T)^{-W}$  as a free module over  $R(T)^W$ .
- (b) If  $V$  is an irreducible  $G$ -vector space. Then

$$[\Lambda] \iota^*[V] = \sum_{w \in W} \text{sgn}(w) [\mathbb{C}_{w \cdot \mu}] \quad (2.3.34)$$

for some weight  $\mu$  in  $\Lambda_T$ . Conversely, for any weight  $\mu$  in  $\Lambda_T$ , there is some irreducible  $G$ -vector space  $V$  such that Equation 2.3.34 holds up to sign.

*Remark.* There is a well-known construction for inducing a representation of  $G$  from a representation of a closed Lie subgroup  $K$  of  $G$ . Suppose  $E$  is a finite-dimensional complex inner product space on which  $K$  acts as unitary transformations. Consider the right  $K$ -action on  $G \times E$  given by  $(g, v) \cdot k = (gk, k^{-1} \cdot v)$ . The orbit space

$$E(G) := G \times_K E$$

is a vector bundle over  $G/K$  with fibers isomorphic to  $E$  (see [63, Prop. 5.4, p. 55]). For each point  $x$  in the fiber over  $\bar{g} := gK$  ( $g \in G$ ), let  $\|x\|_E$  denote the norm of the vector in  $E$  that corresponds to  $x$  under the identification of the fiber over  $\bar{g}$  with  $E$ . Define the  $L^2$ -

norm of a section  $\sigma$  of  $E(G)$  by

$$\|\sigma\|_{L^2}^2 = \int_{G/K} \|\sigma(\bar{g})\|_E^2 d\bar{g},$$

where  $d\bar{g}$  is the quotient measure on  $G/K$  relative to the normalized Haar measures on  $G$  and  $K$ . Let  $\Gamma^2 E(G)$  be the  $L^2$ -closure of the space of smooth sections of  $E(G)$ . The left-regular action of  $G$  on  $\Gamma^2 E(G)$  makes it a  $G$ -vector space. (It is usually infinite-dimensional.) This representation is called the *induced representation* of  $G$  associated to the representation of  $K$  on  $E$ . This induction gives rise to a group homomorphism

$$\iota_* : R(K) \rightarrow \widehat{R}(G) := \prod_{u \in \widehat{G}} \mathbb{Z} \cdot [V_u]. \quad (2.3.35)$$

R. Bott calls the codomain  $\widehat{R}(G)$  as the *formal representation group* of  $G$  and the homomorphism 2.3.35 as the *formal induction*. The group  $\widehat{R}(G)$  contains the ring  $R(G)$ , but the multiplication in  $R(G)$  does not extend to a well-defined multiplication in  $\widehat{R}(G)$ .

**2.3.36 EIGENVALUES OF THE CASIMIR (LAPLACIAN).** Owing to the one-to-one correspondence 2.3.28, we may use  $\Lambda_T \cap K$  to parametrize  $\widehat{G}$  and write the isomorphism 2.2.21 as

$$C^\infty(G, \mathbb{C})_{\text{fin}} \simeq \bigoplus_{\mu \in \Lambda_T \cap K} V(\mu)^* \otimes V(\mu), \quad (2.3.37)$$

where  $V(\mu)$  is an irreducible  $G$ -representation space with highest weight  $\mu$ . Each summand  $V(\mu)^* \otimes V(\mu)$  is an eigenspace of the Casimir  $\Omega$  (see Section 2.2.24). Let  $\Omega(\mu)$  denote the corresponding eigenvalue. One of the motives for reviewing the Weyl formulas was that the eigenvalue  $\Omega(\mu)$  is by which the Laplacian acts on the character of the representation on  $V(\mu)$  (see the end of Section 2.2.24).

**2.3.38 THEOREM.** *Let  $G$  be a compact, simply connected, semisimple Lie group equipped with a bi-invariant metric. The scalar by which the Casimir element  $\Omega$  acts on the irreducible representation space of  $G$  with highest weight  $\mu$  is*

$$\Omega(\mu) = -\|\mu + \rho\|^2 + \|\rho\|^2,$$

where  $\rho$  is half the sum of the positive roots and  $\|\cdot\|$  is the norm on  $\mathfrak{g}^*$  induced by the metric.

*Proof.* The Weyl Character Formula 2.3.29 is an explicit expression for the restriction of  $\chi_\mu$  to  $T$ . So we need to know the differential operator on  $T$  whose action on the restriction  $\chi_\mu|_T$  is equal to  $\Delta_G \chi_\mu$ . If  $G$  is compact, simply connected, and semisimple, then it is known

that, for any smooth class function  $f$  on  $G$ ,

$$(\Delta_G f)|_T = \left( \frac{1}{\mathcal{D}} \circ \Delta_T \circ \mathcal{D} + \|\rho\|^2 \right) f|_T$$

where  $\Delta_T$  is the Laplacian on  $T$  (see [53, pp. 10 and 19]). The compositions appearing in the above equation are to be understood as that of differential operators; for instance,  $\mathcal{D}$  represents the multiplication operator by the function  $\mathcal{D}$ . Substituting  $f$  with  $\chi_\mu$ , we have

$$\Omega(\mu) \chi_\mu|_T = \left( \frac{1}{\mathcal{D}} \circ \Delta_T \circ \mathcal{D} + \|\rho\|^2 \right) \chi_\mu|_T.$$

Applying the Weyl Character Formula 2.3.29, we get

$$\Omega(\mu) \text{Alt}_{\mu+\rho} = \Delta_T \text{Alt}_{\mu+\rho} + \|\rho\|^2 \text{Alt}_{\mu+\rho}. \quad (2.3.39)$$

To calculate  $\Delta_T \text{Alt}_{\mu+\rho}$ , we need to know how  $\Delta_T$  acts on an irreducible character  $\theta_\lambda$  of  $T$  with weight  $\lambda$ . For any  $x$  in  $T$ , we have  $\theta_\lambda(x) = e^{i\langle X_\lambda, H \rangle}$  where  $X_\lambda$  is defined by Equation 2.3.14 and  $H \in \mathfrak{t} \cap \exp^{-1}\{x\}$ . Hence,

$$\Delta_T \theta_\lambda = -\|X_\lambda\| \theta_\lambda = -\|\lambda\| \theta_\lambda.$$

Therefore,

$$\Delta_T \text{Alt}_{\mu+\rho} = -\|\mu + \rho\|^2 \text{Alt}_{\mu+\rho}.$$

Inserting this into Equation 2.3.39 proves the theorem.  $\square$

*Remark.* There is also a Lie algebraic proof; see [62, Prop. 5.28, p. 295].

# 3

## THE VOLUME OF A COMPACT LIE GROUP

As a first application of the asymptotic expansion of the heat trace, we shall verify Harish-Chandra's formula [51, p. 203, Lem. 4] for  $\text{vol}(G)/\text{vol}(T)$ , the ratio of the volume of a compact connected Lie group  $G$  to that of a maximal torus  $T$  in  $G$ , where each volume is measured with respect to the bi-invariant metric generated by an  $\text{Ad}(G)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on the Lie algebra  $\mathfrak{g}$  of  $G$ . The significance of this ratio is its appearance in the Weyl Integration Formula 2.3.19. Harish-Chandra's formula states that

$$\frac{\text{vol}(G)}{\text{vol}(T)} = \prod_{\alpha \in \Phi^+} 2\pi \langle \alpha, \rho \rangle^{-1}. \quad (3.0.1)$$

Here  $\Phi^+$  is the set of (selected) positive roots,  $\rho$  is half the sum of the positive roots, and  $\langle \cdot, \cdot \rangle$  is the inner product on the dual space  $\mathfrak{t}^*$  of the Lie algebra of  $T$  induced by the inner product on  $\mathfrak{g}$ .

An interesting case is when the compact Lie group  $G$  is semisimple. Then we may take the negative of the Killing form 2.1.4 as the  $\text{Ad}(G)$ -invariant inner product on  $\mathfrak{g}$ . Then the right-hand side of Equation 3.0.1 depends only on  $\mathfrak{g}$ .

For the rest of this chapter, we shall assume that the compact connected Lie group  $G$  is semisimple and simply connected; there is no loss of generality, as far as  $\text{vol}(G)/\text{vol}(T)$  is concerned, for the following reasons. By the general theory of compact connected Lie groups, every compact connected Lie group is isomorphic to

$$G \simeq (R \times S)/F \quad (3.0.2)$$

where  $R$  is a torus,  $S$  is a compact, connected, simply connected, semisimple Lie group, and  $F$  is a finite abelian subgroup of  $R \times S$  (see [62, Thm. 4.29, p. 250]). So  $F$  is contained in a maximal torus



$\tilde{T}$  of  $R \times S$ . Then  $\tilde{T}/F$  is a maximal torus of  $(R \times S)/F$ ; let  $T$  be the corresponding maximal torus in  $G$  under the isomorphism 3.0.2. The Lie groups  $G$ ,  $R \times S$ , and  $(R \times S)/F$  all have isomorphic Lie algebras. So the  $\text{Ad}(G)$ -invariant inner product on  $\mathfrak{g}$  induces a bi-invariant metric on the Lie groups  $G$ ,  $R \times S$ ,  $(R \times S)/F$ , and their respective maximal tori. Their Riemannian volumes satisfy

$$\frac{\text{vol}(R \times S)}{\text{vol}(\tilde{T})} = \frac{\text{vol}((R \times S)/F)}{\text{vol}(\tilde{T}/F)} = \frac{\text{vol}(G)}{\text{vol}(T)}.$$

Hence, we may assume that  $F$  is trivial so that  $G \simeq R \times S$ . Now the maximal torus  $\tilde{T}$  of  $R \times S$  is of the form  $R \times T_S$ , where  $T_S$  is a maximal torus of  $S$ . Hence,

$$\frac{\text{vol}(R \times S)}{\text{vol}(\tilde{T})} = \frac{\text{vol}(R \times S)}{\text{vol}(R \times T_S)} = \frac{\text{vol}(S)}{\text{vol}(T_S)}.$$

This shows that we may as well assume that  $G = S$ , that is,  $G$  is semisimple and simply connected.

Derivations of the formula 3.0.1 can also be found in [34, 38, 40, 72]; among these, H. D. Fegan [38] uses the heat trace approach. Our calculation differs from H. D. Fegan's, in that we use the Euler-Maclaurin Formula and the Weyl Integration Formula as our main tools for calculation.

### 3.1 THE EULER-MACLAURIN FORMULA

Let  $f$  be a smooth function on the real line. The Euler-Maclaurin Formula [36, 37, 73] relates a sum  $\sum_{x=0}^n f(x)$  to the integral  $\int_0^n f(x) dx$ . We are interested in this formula since the heat trace is a sum of the type  $\sum_{x=0}^n f(x)$  but with  $n = \infty$ . The formula states, for  $N \in \mathbb{N}$ ,

$$\sum_{x=0}^n f(x) = \int_0^n f(x) dx + \sum_{q=1}^N (-1)^q \frac{B_q}{q!} (f^{(q-1)}(n) - f^{(q-1)}(0)) + R_N, \quad (3.1.1)$$

where the coefficients  $B_q$  are the Bernoulli numbers defined by the power series

$$\text{td}(x) := \frac{x}{1 - e^{-x}} = \sum_{q=0}^{\infty} (-1)^q \frac{B_q}{q!} x^q,$$

and  $R_N$  is the remainder term, which can be estimated by

$$|R_N| \leq \frac{2\zeta(N)}{(2\pi)^N} \int_0^n |f^{(N)}(x)| dx. \quad (3.1.2)$$

Here  $\zeta$  is the Riemann zeta function. Suppose  $\sum_{k=0}^{\infty} f(k)$  exists and  $\lim_{x \rightarrow \infty} f^{(q)}(x) = 0$  for all  $q \in \mathbb{N}$ . Then we may take  $n = \infty$  in the

formula 3.1.1, which gives us

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(x) dx + \sum_{q=1}^N (-1)^q \frac{B_q}{q!} f^{(q-1)}(0) + R_N. \quad (3.1.3)$$

If furthermore  $R_N \rightarrow 0$  as  $N \rightarrow \infty$  (for instance, when  $f$  is a polynomial), we have

$$\sum_{k=0}^{\infty} f(k) = \text{td} \left( \frac{\partial}{\partial h} \right) \Big|_{h=0} \int_{-h}^{\infty} f(x) dx. \quad (3.1.4)$$

(This expression for the Euler-Maclaurin Formula first appeared in [81].)

The following Lemma can be proved using the Euler-Maclaurin Formula. We shall make use of it later on.

**3.1.5 LEMMA.** *Let  $A$  and  $B$  be real numbers with  $A > 0$ , and let  $m$  be any nonnegative integer. Let*

$$f_t(x) = x^{2m} e^{-t(Ax^2+Bx)}.$$

*Consider the sum, for  $t > 0$ ,*

$$S(t) = \sum_{x=0}^{\infty} f_t(x).$$

*Then,*

$$t^{m+1/2} S(t) = \int_0^{\infty} f_t(x) dx + O(t^{1/2})$$

*for  $t \rightarrow 0+$ .*

*Proof.* By Equation 3.1.3 with  $N = 1$ ,

$$S(t) = \int_0^{\infty} f_t(x) dx - B_1 f_t(0) + R_1.$$

By the estimate 3.1.2, the absolute value of  $R_1$  is bounded by, up to a constant factor,

$$\int_0^{\infty} \left| 2mx^{2m-1} e^{-t(Ax^2+Bx)} - t(2Ax^{2m+1} + Bx^{2m}) e^{-t(Ax^2+Bx)} \right| dx.$$

By a change of variable of the type  $x \mapsto \alpha x + \beta$ ,  $\alpha > 0$ , the above integral can be recast in the following form:

$$\int_{\alpha}^{\infty} \left| P_1(x) e^{-tx^2} + tP_2(x) e^{-tx^2} \right| dx,$$

where  $P_1(x)$  and  $P_2(x)$  are polynomials of degree  $2m-1$  and  $2m+1$ , respectively. This integral is bounded by, up to a constant factor,

$$\int_{-\infty}^{\infty} |P_1(x)| e^{-tx^2} dx + t \int_{-\infty}^{\infty} |P_2(x)| e^{-tx^2} dx. \quad (3.1.6)$$

It is known that, for a nonnegative integer  $n$ ,

$$\int_{-\infty}^{\infty} |x|^n e^{-tx^2} dx = \frac{1}{t^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right), \quad (3.1.7)$$

where  $\Gamma$  denotes the Gamma function. Hence, we see that the integral 3.1.6 is bounded by a quantity that is of  $O(t^{-m})$  for  $t \rightarrow 0+$ . Meanwhile, the integral  $\int_0^{\infty} f_t(x) dx$  is of  $O(t^{-m-1/2})$ . Hence,

$$t^{m+1/2}S(t) = \int_0^{\infty} f_t(x) dx + O(t^{1/2}). \quad \square$$

### 3.2 AN EXAMPLE: $SU(2)$

3.2.1 Before we launch into general arguments, we take  $SU(2)$  as a concrete example to illustrate our strategy.

3.2.2 SETUP. Let  $G$  be  $SU(2)$ , the set of  $2 \times 2$  unitary matrices of determinant 1. Its Lie algebra  $\mathfrak{g}$  is  $\mathfrak{su}(2)$ , the set of  $2 \times 2$  complex matrices that are traceless and antihermitian. It is known that  $G$  is diffeomorphic to the 3-sphere (see [34, § 1.2.B]); hence, it is compact, connected, and simply connected. As we shall shortly see, the Killing form  $\kappa$  on  $\mathfrak{g}$  is negative definite; so  $G$  is semisimple.

For the maximal torus  $T$  in  $G$ , we take

$$T = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} : t \in \mathbb{R} \right\} \simeq U(1).$$

We pick the following matrices as the basis vectors for  $\mathfrak{g}$ :

$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The vector  $H$  lies in the Lie algebra  $\mathfrak{t}$  of  $T$ .

The matrices  $[\text{ad}_X]$ ,  $[\text{ad}_Y]$ , and  $[\text{ad}_H]$  for the adjoint representation of the basis vectors  $X$ ,  $Y$ , and  $H$ , respectively, are given by

$$[\text{ad}_X] = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot \end{pmatrix}, \quad [\text{ad}_Y] = \begin{pmatrix} \cdot & \cdot & -2 \\ \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot \end{pmatrix}, \quad [\text{ad}_H] = \begin{pmatrix} \cdot & 2 & \cdot \\ -2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

Here the dots represent 0. From this, we can calculate the matrix  $[\eta]$  for the bilinear form  $\eta = -\kappa$  on  $\mathfrak{g}$ ; it turns out that

$$[\eta] = \begin{pmatrix} 8 & \cdot & \cdot \\ \cdot & 8 & \cdot \\ \cdot & \cdot & 8 \end{pmatrix}. \quad (3.2.3)$$

We take the bi-invariant metric induced by  $\eta$  as the metric for  $G$ .

3.2.4 ROOTS. The complexification of  $\mathfrak{su}(2)$  is  $\mathfrak{sl}(2, \mathbb{C})$ . The standard basis for  $\mathfrak{sl}(2, \mathbb{C})$  is given by the following matrices:

$$\begin{aligned} J^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(-iX + Y), \\ J^- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(-iX - Y), \\ J^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -iH. \end{aligned}$$

The basis elements satisfy the Lie bracket relations

$$[J^+, J^-] = J^0, \quad (3.2.5)$$

$$[J^0, J^\pm] = \pm 2J^\pm. \quad (3.2.6)$$

Equation 3.2.6 shows that there are two complex roots  $\pm\mu$ , which are characterized by

$$\pm\mu(H) = \pm i\mu(J^0) = \pm 2i.$$

Denoting the corresponding real roots as  $\pm\alpha$ , we have

$$\pm\alpha(H) = \pm 2.$$

The Weyl group is the cyclic group of order 2 generated by  $\sigma : \alpha \mapsto -\alpha$ . If we take  $\alpha$  as the basis for the dual space  $\mathfrak{t}^*$  of  $\mathfrak{t}$ , thereby identifying  $\mathfrak{t}^*$  with the real line  $\mathbb{R}$ , then  $\sigma$  is the reflection with respect to the origin. We choose the closed half-line  $[0, \infty[$  as the fundamental Weyl chamber  $K$ . This amounts to choosing  $\alpha$  as the positive root.

**3.2.7 DOMINANT WEIGHTS.** The dual vector  $X_\alpha$  of  $\alpha$  is characterized by

$$\alpha(H) = \langle X_\alpha, H \rangle = 2. \quad (3.2.8)$$

This implies  $X_\alpha = H/4$ . Since  $\alpha(X_\alpha) = \langle X_\alpha, X_\alpha \rangle = 1/2$ , we have

$$\langle \alpha, \alpha \rangle = \langle X_\alpha, X_\alpha \rangle = \frac{1}{2}. \quad (3.2.9)$$

There is only one fundamental dominant weight, which is

$$\rho := \frac{1}{2}\alpha.$$

We have

$$\langle \rho, \rho \rangle = \frac{1}{4}\langle \alpha, \alpha \rangle = \frac{1}{8}. \quad (3.2.10)$$

Since  $SU(2)$  is simply connected, the weight lattice  $\Lambda_T$  is spanned by  $\rho$  (see Section 2.3.15). The set of dominant weights is

$$\Lambda_T \cap K = \{\ell\rho \mid \ell = 0, 1, 2, \dots\}.$$

**3.2.11 HEAT TRACE.** Let  $V(\ell)$  denote the highest weight module with highest weight  $\ell\rho$ . Then, by the isomorphism 2.3.37,

$$C^\infty(G, \mathbb{C})_{\text{fin}} \simeq \bigoplus_{\ell=0}^{\infty} V(\ell) \otimes V(\ell)^*.$$

By the Weyl Dimension Formula 2.3.31,

$$\dim V(\ell) = \frac{\langle \alpha, (\ell+1)\rho \rangle}{\langle \alpha, \rho \rangle} = \ell + 1. \quad (3.2.12)$$

The value of the Casimir on  $V(\ell)$  is (Theorem 2.3.38):

$$\Omega(\ell) = -(\ell + 1)^2 \|\rho\|^2 + \|\rho\|^2 = -\frac{(\ell + 1)^2}{8} + \frac{1}{8}. \quad (3.2.13)$$

Therefore, the heat trace is

$$\mathrm{tr}(e^{t\Delta_G}) = \sum_{\ell=0}^{\infty} \dim V(\ell)^2 e^{t\Omega(\ell)} = e^{t/8} \sum_{\ell=0}^{\infty} (\ell + 1)^2 e^{-t(\ell+1)^2/8}. \quad (3.2.14)$$

For the volume of  $G$ , we are only interested in the leading term of the asymptotic expansion of the heat trace. Thus, we only need to calculate

$$Z(t) := \sum_{\ell=0}^{\infty} (\ell + 1)^2 e^{-t(\ell+1)^2/8}.$$

Define the polynomial  $d(\ell)$  in  $\ell$  by

$$d(\ell) = \ell.$$

Then,

$$Z(t) = \sum_{n=0}^{\infty} d(n)^2 e^{-tn^2/8}.$$

**3.2.15 ASYMPTOTIC EXPANSION.** Set

$$f_t(x) := x^2 e^{-tx^2/8}.$$

Then

$$Z(t) = \sum_{x=0}^{\infty} f_t(x).$$

By the Euler-Maclaurin Formula 3.1.3,

$$Z(t) = \int_0^{\infty} f_t(x) dx + \frac{1}{2}f(0) - \sum_{p=1}^N \frac{B_{p+1}}{(p+1)!} f_t^{(p)}(0) + R_N. \quad (3.2.16)$$

We are only interested in the leading term of the asymptotic expansion of  $Z(t)$ . By Lemma 3.1.5,

$$t^{3/2}Z(t) = t^{3/2} \int_0^{\infty} f_t(x) dx + O(t) \quad (3.2.17)$$

for  $t \rightarrow 0+$ .

**3.2.18 SYMMETRY.** The symmetry under the Weyl reflection now comes into play. The function  $f_t(x)$  is invariant under the reflection  $x \mapsto -x$ . So

$$Z(t) = \frac{1}{2} \sum_{x \in \mathbb{Z}} f_t(x).$$

And Equation 3.2.17 can be rewritten as

$$t^{3/2}Z(t) = t^{3/2}I(t) + O(t), \quad (3.2.19)$$

where

$$I(t) := \frac{1}{2} \int_{-\infty}^{\infty} f_t(x) dx. \quad (3.2.20)$$

This integral is essentially a Gaussian integral, which can be easily computed (See Equation 3.1.7):

$$I(t) = \frac{4\sqrt{2\pi}}{t^{3/2}}.$$

So Equation 3.2.19 gives

$$t^{3/2}Z(t) = 4\sqrt{2\pi} + O(t). \quad (3.2.21)$$

3.2.22 VOLUME OF G AND T. Owing to Weyl's law 1.0.6, the asymptotic equality 3.2.21 yields

$$\text{vol}(G) = 32\sqrt{2\pi}^2. \quad (3.2.23)$$

The volume of T is given by

$$\text{vol}(T) = \int_0^{2\pi} \langle H_\alpha, H_\alpha \rangle^{1/2} dt = 2\pi \langle H_\alpha, H_\alpha \rangle^{1/2},$$

where  $H_\alpha$  is the coroot in  $\mathfrak{t}$  associated to  $\alpha$ . (We shall prove this in Lemma 3.3.9(b).) The coroot  $H_\alpha$  is characterized by

$$\lambda(H_\alpha) = \frac{1}{2}\alpha(H_\alpha) = 1.$$

Thus,  $H_\alpha = H$ . As the matrix 3.2.3 shows,  $\langle H, H \rangle = 8$ . So the volume of T is

$$\text{vol}(T) = 4\sqrt{2\pi}.$$

Combining with Equation 3.2.23, we have

$$\frac{\text{vol}(G)}{\text{vol}(T)} = 8\pi. \quad (3.2.24)$$

This agrees with Harish-Chandra's formula 3.0.1, which gives

$$\frac{\text{vol}(G)}{\text{vol}(T)} = 2\pi \langle \alpha, \alpha/2 \rangle^{-1} = 4\pi \langle \alpha, \alpha \rangle^{-1} = 8\pi,$$

where we have used Equation 3.2.9 in the last step.

### 3.3 THE GENERAL CASE

3.3.1 We now consider the general case. Our aim is to prove Equation 3.3.16 for any compact connected Lie group G. We shall follow in outline the calculation presented in Section 3.2. The only new ingredient that will appear is the Weyl Integration Formula.

We adopt the following notation for the dimension of G, T, and

$G/T$ :

$$n := \dim G, \quad r := \dim T, \quad 2m := \dim G - \dim T = n - r.$$

We can read from the root space decomposition 2.3.13 that  $2m$  is equal to the number of roots of  $G$ . Since half of the roots are positive (see page 31), we have

$$m = |\Phi^+|. \quad (3.3.2)$$

3.3.3 As we explained in the beginning of this chapter, we may assume that  $G$  is compact, connected, simply connected, and semi-simple. In this case, there is a one-to-one correspondence between the unitary dual  $\widehat{G}$  of  $G$  and the set  $\Lambda_T \cap K$  of dominant weights of  $G$ . For each  $\lambda \in \Lambda_T \cap K$ , let  $V(\lambda)$  denote a highest weight module with highest weight  $\lambda$ . By Equation 2.2.26 and Theorem 2.3.38, we have, for the heat trace,

$$\mathrm{tr}(e^{t\Delta_G}) = \sum_{\lambda \in \Lambda_T \cap K} \dim V(\lambda)^2 e^{t\Omega(\lambda)},$$

where

$$\Omega(\lambda) = -\|\lambda + \rho\|^2 + \|\rho\|^2.$$

The dimension of  $V(\lambda)$  can be calculated by the Weyl Dimension Formula 2.3.31. If we let

$$d(\lambda) := \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \lambda \rangle}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}, \quad (3.3.4)$$

then the Weyl Dimension Formula takes the form

$$\dim V(\lambda) = d(\lambda + \rho).$$

So the heat trace can be expressed as

$$\mathrm{tr}(e^{t\Delta_G}) = e^{t\|\rho\|^2} \sum_{\lambda \in \Lambda_T \cap K} d(\lambda + \rho)^2 e^{-t\|\lambda + \rho\|^2}. \quad (3.3.5)$$

The leading term of the asymptotic expansion of the heat trace 3.3.5 is identical with that of

$$Z(t) := \sum_{\lambda \in \Lambda_T \cap K} d(\lambda + \rho)^2 e^{-t\|\lambda + \rho\|^2} = \sum_{\lambda \in (\Lambda_T \cap K) + \rho} d(\lambda)^2 e^{-t\|\lambda\|^2}. \quad (3.3.6)$$

Note that the shifted index set  $(\Lambda_T \cap K) + \rho$  is the set of the weights that lie in the interior of the Weyl chamber. (This can be deduced from the properties of the fundamental weights that we listed in Section 2.3.15.) Hence,

$$Z(t) = \sum_{\lambda \in \Lambda_T \cap K^\circ} d(\lambda)^2 e^{-t\|\lambda\|^2}. \quad (3.3.7)$$

But since the restriction of the function  $d$  to the boundary of the Weyl chamber is zero, we may replace the index set  $\Lambda_T \cap K^\circ$  of the

sum 3.3.7 with  $\Lambda_T \cap K$  and write

$$Z(t) = \sum_{\lambda \in \Lambda_T \cap K} d(\lambda)^2 e^{-t\|\lambda\|^2}. \quad (3.3.8)$$

3.3.9 LEMMA. Let  $\mu_{\mathfrak{t}^*}$  denote the Lebesgue measure on  $\mathfrak{t}^*$  induced by the inner product  $\langle \cdot, \cdot \rangle$ . Let  $\text{vol}(P)$  be the volume of the fundamental parallelepiped formed by the fundamental weights of  $G$  in  $\mathfrak{t}^*$ .

(a) For  $t \rightarrow 0+$ , we have

$$t^{n/2} Z(t) = t^{n/2} I(t) + O(t), \quad (3.3.10)$$

where

$$I(t) = \int_K d(\lambda)^2 e^{-t\|\lambda\|^2} \frac{\mu_{\mathfrak{t}^*}(\lambda)}{\text{vol}(P)}. \quad (3.3.11)$$

(b) The volume of the maximal torus is related to  $\text{vol}(P)$  by

$$\text{vol}(T) = (2\pi)^r \text{vol}(P)^{-1}. \quad (3.3.12)$$

*Proof.* (a) Denote the set of fundamental dominant weights by

$$\{\lambda_1, \dots, \lambda_r\}.$$

This is a basis for  $\mathfrak{t}^*$ . Let us use  $(x_1, \dots, x_r)$  to denote the components of a vector in  $\mathfrak{t}^*$  with respect to this basis. Then  $Z(t)$  is of the form

$$Z(t) = \sum_{x_1=0}^{\infty} \cdots \sum_{x_r=0}^{\infty} d(x_1, \dots, x_r)^2 e^{-tq(x_1, \dots, x_r)},$$

where  $d$  and  $q$  are homogeneous polynomials of degree  $m$  and  $2$ , respectively. In terms for the variables  $(x_1, \dots, x_r)$ , the integral 3.3.11 takes the form

$$I(t) = \int_0^{\infty} \cdots \int_0^{\infty} d(x_1, \dots, x_r)^2 e^{-tq(x_1, \dots, x_r)} dx_1 \cdots dx_r.$$

Applying Lemma 3.1.5 iteratively to  $Z(t)$  proves the assertion.

(b) Because  $G$  is simply connected, the coroot lattice  $\Lambda_{\text{coroot}}$  coincides with the lattice  $\Lambda_e := \mathfrak{t} \cap \exp^{-1}\{e\}$ . By duality, the lattice  $\Lambda_{\mathfrak{g}}$  of algebraically integral weights coincides with the lattice  $\Lambda_T$  of analytically integral weights. Thus, the simple coroots  $\{H_{\alpha_i}\}_{i=1}^r$  form a basis for  $\Lambda_e$ , and the corresponding fundamental weights  $\{\lambda_i\}_{i=1}^r$  form a basis for  $\Lambda_T$ .

Let  $Q$  be the fundamental parallelepiped in  $\mathfrak{t}$  formed by the simple coroots. Let  $\mu_{\mathfrak{t}}$  be the Lebesgue measure on  $\mathfrak{t}$  induced by the inner product  $\langle \cdot, \cdot \rangle$ . Since  $\exp(2\pi H_{\alpha}) = e$  for any root  $\alpha$ , we have

$$\text{vol}(T) = (2\pi)^r \text{vol}(Q). \quad (3.3.13)$$

Meanwhile, because the lattice  $\Lambda_e$  and  $\Lambda_T$  are dual to each other, we have

$$\text{vol}(P) = \text{vol}(Q)^{-1}. \quad (3.3.14)$$

This proves the assertion.  $\square$



3.3.15 THEOREM. Let  $G$  be a compact connected Lie group and  $T$  its maximal torus. Let  $\mathfrak{g}$  and  $\mathfrak{t}$  be their Lie algebras, respectively. Let  $\langle \cdot, \cdot \rangle$  denote an  $\text{Ad}(G)$ -invariant inner product on  $\mathfrak{g}$  and also the induced inner product on the dual space  $\mathfrak{g}^*$ . The Riemannian volume of  $G$  and  $T$  with respect to the bi-invariant metric generated by  $\langle \cdot, \cdot \rangle$  are related by

$$\frac{\text{vol}(G)}{\text{vol}(T)} = \prod_{\alpha \in \Phi^+} 2\pi \langle \alpha, \rho \rangle^{-1}, \quad (3.3.16)$$

where  $\Phi^+$  is the set of the selected positive roots of  $G$  and  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ .

*Proof.* We point out once again that  $G$  may be assumed to be simply connected and semisimple (as explained on pages 39–40). The set of roots and the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{t}^*$  are preserved under the action of the Weyl group  $W$ . This implies that  $|d(\lambda)|$  and  $\|\lambda\|$  that appear in the integral 3.3.11 are preserved under the  $W$ -action; so it is possible to extend the domain of integration to all of  $\mathfrak{t}^*$  as follows:

$$\begin{aligned} I(\mathfrak{t}) &= \frac{1}{|W|} \int_{\mathfrak{t}^*} d(\lambda)^2 e^{-t\|\lambda\|^2} \frac{\mu_{\mathfrak{t}^*}(\lambda)}{\text{vol}(\mathfrak{P})} \\ &= \frac{1}{|W|} \frac{\text{vol}(T)}{(2\pi)^r} \int_{\mathfrak{t}^*} d(\lambda)^2 e^{-t\|\lambda\|^2} \mu_{\mathfrak{t}^*}(\lambda). \end{aligned} \quad (3.3.17)$$

We have used Equation 3.3.13 for the last equality.

We wish to change the domain of integration from  $\mathfrak{t}^*$  to  $\mathfrak{t}$  via the isomorphism

$$\begin{aligned} \mathfrak{t}^* &\rightarrow \mathfrak{t}, \\ \lambda &\mapsto X_\lambda, \end{aligned}$$

where  $X_\lambda$  is defined by the equation

$$\lambda(H) = \langle X_\lambda, H \rangle, \quad \forall H \in \mathfrak{t}.$$

This isomorphism is precisely through which the inner product on  $\mathfrak{t}$  was transferred to  $\mathfrak{t}^*$ ; in particular, the Jacobian determinant of this isomorphism is 1. Next, we have

$$d(\lambda)^2 = \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \lambda \rangle^2}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2} = \frac{\prod_{\alpha \in \Phi^+} \alpha(X_\lambda)^2}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2}, \quad (3.3.18)$$

and

$$e^{-t\|\lambda\|^2} = e^{-t\|X_\lambda\|^2}.$$

Thus,

$$I(\mathfrak{t}) = \frac{1}{|W|} \frac{\text{vol}(T)}{(2\pi)^r} \left( \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^{-2} \right) \int_{\mathfrak{t}} \left( \prod_{\alpha \in \Phi^+} \alpha(X)^2 \right) e^{-t\|X\|^2} \mu_{\mathfrak{t}}(X), \quad (3.3.19)$$

where  $\mu_{\mathfrak{t}}$  is the Lebesgue measure on  $\mathfrak{t}$  induced by the inner product.

Recall the Weyl Integration Formula 2.3.23, which states that, for

any  $\text{ad}(\mathfrak{g})$ -invariant function  $f$  on  $\mathfrak{g}$ ,

$$\int_{\mathfrak{g}} f(X) \mu_{\mathfrak{g}}(X) = \frac{1}{|W|} \frac{\text{vol}(G)}{\text{vol}(T)} \int_t f(X) \left( \prod_{\alpha \in \Phi^+} \alpha(X)^2 \right) \mu_t(X). \quad (3.3.20)$$

The measure  $\mu_{\mathfrak{g}}$  on  $\mathfrak{g}$  is again the Lebesgue measure induced by the inner product. Inserting  $f(X) = e^{-t\|X\|^2}$  into Equation 3.3.20 and rearranging the terms, we get

$$\int_t e^{-t\|X\|^2} \left( \prod_{\alpha \in \Phi^+} \alpha(X)^2 \right) \mu_t(X) = |W| \frac{\text{vol}(T)}{\text{vol}(G)} \int_{\mathfrak{g}} f(X) \mu_{\mathfrak{g}}(X).$$

Applying this equation to the right-hand side of Equation 3.3.19, we get

$$I(t) = \frac{\text{vol}(T)^2}{(2\pi)^r \text{vol}(G)} \left( \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^{-2} \right) \int_{\mathfrak{g}} e^{-t\|X\|^2} \mu_{\mathfrak{g}}(X). \quad (3.3.21)$$

The last integral is just a Gaussian integral:

$$\int_{\mathfrak{g}} e^{-t\|X\|^2} \mu_{\mathfrak{g}}(X) = \left( \frac{\pi}{t} \right)^{n/2}.$$

Hence,

$$I(t) = \frac{\text{vol}(T)^2}{(2\pi)^r \text{vol}(G)} \left( \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^{-2} \right) \left( \frac{\pi}{t} \right)^{n/2}. \quad (3.3.22)$$

Inserting this into Equation 3.3.10 and invoking Weyl's law 1.0.6, we get

$$\frac{\text{vol}(G)^2}{\text{vol}(T)^2} = (2\pi)^{2m} \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^{-2}.$$

The inner product between two positive roots is nonnegative. (This owes to the definition of  $\Phi^+$ ; see page 30.) Hence,

$$\frac{\text{vol}(G)}{\text{vol}(T)} = (2\pi)^m \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^{-1}.$$

By Equation 3.3.2, we have

$$\frac{\text{vol}(G)}{\text{vol}(T)} = \prod_{\alpha \in \Phi^+} 2\pi \langle \alpha, \rho \rangle^{-1}. \quad \square$$

# 4

## THE HEAT KERNEL OF THE LAPLACIAN ON A COMPACT LIE GROUP

WE now come to one of our main objectives — the calculation of the asymptotic expansion for the heat kernel of the Laplacian  $\Delta_G$  on a compact connected Lie group  $G$  that is endowed with a bi-invariant metric  $\langle \cdot, \cdot \rangle$ . Our aim is to show that one can quickly obtain the asymptotic expansion using Lie algebra methods. The key ingredient is the Duflo isomorphism [32]

$$\text{Duf} : S(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathcal{Z}(\mathfrak{g}),$$

which is an algebra homomorphism from the  $\text{ad}(\mathfrak{g})$ -invariant part of the symmetric algebra of  $\mathfrak{g}$  (elements of  $\text{ad}(\mathfrak{g})$  act as inner derivations) to the center of the universal enveloping algebra of  $\mathfrak{g}$ . The domain  $S(\mathfrak{g})^{\mathfrak{g}}$  can be identified with the space of constant coefficient differential operators on  $\mathfrak{g}$  that are invariant under the  $\text{Ad}(G)$ -action, and the image  $\mathcal{Z}(\mathfrak{g})$  can be identified with the space of bi-invariant differential operators on  $G$ . In  $S(\mathfrak{g})^{\mathfrak{g}}$  is the Laplacian  $\Delta_{\mathfrak{g}}$  of the Euclidean space  $\mathfrak{g}$ , and in  $\mathcal{Z}(\mathfrak{g})$  is the Laplacian  $\Delta_G$  of the curved space  $G$ .

The main idea is this: Find a relation between  $\text{Duf}(\Delta_{\mathfrak{g}})$  and  $\Delta_G$ , and use that to deduce the asymptotic expansion for the heat kernel of  $\Delta_G$ . What makes this a plausible route? Assume for the moment that the relationship between  $\text{Duf}(\Delta_{\mathfrak{g}})$  and  $\Delta_G$  is simple enough so that the heat kernel of  $\Delta_G$  is a minor modification of the heat kernel of  $\text{Duf}(\Delta_{\mathfrak{g}})$ . The heat kernel of  $\text{Duf}(\Delta_{\mathfrak{g}})$  is the integral kernel of the operator  $e^{t \text{Duf}(\Delta_{\mathfrak{g}})}$ . Recall that the Duflo isomorphism is an algebra isomorphism. So, heuristically, we expect  $e^{t \text{Duf}(\Delta_{\mathfrak{g}})}$  to be equal to “ $\text{Duf}(e^{t \Delta_{\mathfrak{g}}})$ ”. Now the integral kernel of  $e^{t \Delta_{\mathfrak{g}}}$  is well-known to be the Gaussian kernel. So our expectation is that, by understanding

the geometric meaning of the Duflo isomorphism, we would be able to figure out its effect on the Gaussian kernel and deduce from it the integral kernel of “ $\text{Duf}(e^{t\Delta_{\mathfrak{g}}})$ ”, or rather, for  $e^{t\text{Duf}(\Delta_{\mathfrak{g}})}$ , which is essentially  $e^{t\Delta_{\mathfrak{g}}}$ . This last statement can be made more precise using the Kashiwara-Vergne conjecture [60] (now proved), which extends the Duflo isomorphism to incorporate invariant distributions with compact support. In this dissertation, however, we shall rely on more elementary means.

*Remark.* When we speak of a *differential operator on a manifold*  $M$ , we shall assume that its domain is  $C^\infty(M)$ , unless explicitly stated otherwise.

## 4.1 THE DUFLO ISOMORPHISM

**4.1.1 INVARIANT DIFFERENTIAL OPERATORS.** Let  $A$  be any Lie group. Suppose  $A$  acts smoothly on a manifold  $M$  on the left. Let  $g$  be an element of  $A$ . For a smooth function  $f$  on  $M$ , we define the smooth function  $g \cdot f$  on  $M$  by

$$(g \cdot f)(x) = f(g^{-1} \cdot x). \quad (4.1.2)$$

For a differential operator  $D$  on  $M$ , we define the differential operator  $g \cdot D$  on  $M$  by

$$(g \cdot D)f = g \cdot (D(g^{-1} \cdot f)). \quad (4.1.3)$$

The differential operator  $D$  is said to be *invariant* if  $(g \cdot D) = D$  holds for all  $g$  in  $A$ .

Now let  $M = A$ , and consider the action of  $A$  on itself by left translations. In this case, the invariant differential operators on  $A$  are said to be *left-invariant*. Let  $D(A)$  be the space the left-invariant differential operators on  $A$ . Let  $\mathfrak{a}$  be the Lie algebra of  $A$ . The construction of the universal enveloping algebra  $\mathcal{U}(\mathfrak{a})$  of  $\mathfrak{a}$  and its identification with  $D(A)$ , which we reviewed in Section 2.2.10, still applies here, and we have an algebra isomorphism

$$\begin{aligned} \tau: \quad \mathcal{U}(\mathfrak{a}) &\rightarrow D(A), \\ X_1 \otimes \cdots \otimes X_k &\mapsto \tilde{X}_1 \circ \cdots \circ \tilde{X}_k. \end{aligned} \quad (4.1.4)$$

Here  $\tilde{X}$  denotes the left-invariant vector field on  $A$  generated by  $X$  in  $\mathfrak{a}$ .

Suppose  $D$  is an invariant differential operator on  $A$ . Let  $D_g$  denote the value of  $D$  at  $g$  in  $A$ ; in other words, for a smooth function  $f$  on  $A$ ,

$$D_g f = (Df)(g).$$

Since Equation 4.1.3 holds for  $D$ , we have

$$g^{-1} \cdot (Df) = D(g^{-1} \cdot f).$$

Evaluating both sides of this equation at the identity, we get

$$(Df)(g) = (D(g \cdot f))(e).$$

Put in another way,

$$D_g f = D_e(g^{-1} \cdot f).$$

In short,  $D$  is completely determined by its value at the identity.

Now  $D_e : C^\infty(A) \rightarrow \mathbb{R}$  is a distribution on  $A$  that is supported on  $\{e\}$ . So we have a linear map from the space of left-invariant differential operators on  $A$  to the space of distributions on  $A$  with support  $\{e\}$ ;

$$\begin{aligned} \varepsilon : D(A) &\rightarrow \mathcal{E}'_e(A), \\ D &\mapsto D_e. \end{aligned} \quad (4.1.5)$$

This is, in fact, an algebra isomorphism. The injectivity is clear. The surjectivity follows from a standard result in distribution theory, namely, that a distribution with point support is a differential operator (see [52, Prop. 1, p. 242]). To prove the multiplicativity, we need to show that

$$(\widetilde{X}\widetilde{Y})_e = \widetilde{X}_e * \widetilde{Y}_e \quad (4.1.6)$$

where  $*$  denotes the convolution product. The pairing of  $\widetilde{X}_e * \widetilde{Y}_e$  with  $f$  in  $C^\infty(A)$  is defined as

$$\langle \widetilde{X}_e * \widetilde{Y}_e, f \rangle := \langle \widetilde{X}_e \otimes \widetilde{Y}_e, \hat{f} \rangle,$$

where  $\hat{f}$  is a smooth function on  $A \times A$  such that

$$\hat{f}(x, y) = f(xy) = \ell_x^* f(y),$$

and the  $i$ th factor ( $i = 1, 2$ ) of  $\widetilde{X}_e \otimes \widetilde{Y}_e$  pairs with  $\hat{f}$  by viewing it as a function that depends only on the  $i$ th variable; thus,  $\langle \widetilde{X}_e \otimes \widetilde{Y}_e, \hat{f} \rangle = \langle \widetilde{X}_e, F_{Y,f} \rangle$  where  $F_{Y,f}$  is a smooth function on  $A$  whose value at  $x \in A$  is

$$F_{Y,f}(x) := \langle \widetilde{Y}_e, \ell_x^* f \rangle = \left. \frac{d}{dt} \right|_0 f(x \exp(tY)). \quad (4.1.7)$$

Thus,

$$\begin{aligned} \langle \widetilde{X}_e \otimes \widetilde{Y}_e, \hat{f} \rangle &= \langle \widetilde{X}_e, F_{Y,f} \rangle = \left. \frac{d}{dt} \right|_0 F_{Y,f}(\exp(tX)) \\ &= \left. \frac{d}{dt} \right|_0 \left. \frac{d}{ds} \right|_0 f(\exp(tX) \exp(sY)) = \langle (\widetilde{X}\widetilde{Y})_e, f \rangle. \end{aligned}$$

This proves Equation 4.1.6, which implies that the linear isomorphism 4.1.5 is multiplicative.

Composing the isomorphisms 4.1.4 and 4.1.5, we get an algebra isomorphism

$$\delta := \varepsilon \circ \tau : \mathcal{U}(\mathfrak{a}) \rightarrow \mathcal{E}'_e(A). \quad (4.1.8)$$

This gives us a second identification for  $\mathcal{U}(\mathfrak{a})$ , namely, as the algebra of distributions on  $A$  with point support on the identity. Applying this to the case  $A = G$ , we see that  $\mathcal{U}(\mathfrak{a}) = \mathcal{U}(\mathfrak{g})$  can be viewed as

the algebra of distributions on  $G$  with support  $\{e\}$ . Applying to the case  $A = \mathfrak{g}$ , we see that  $\mathcal{U}(\mathfrak{a}) = S(\mathfrak{g})$  can be viewed as the algebra of distributions on  $\mathfrak{g}$  with support  $\{0\}$ .

**4.1.9 POINCARÉ-BIRKHOFF-WITT ISOMORPHISM.** Let  $T(\mathfrak{g})$  be the tensor algebra of  $\mathfrak{g}$ . Let  $I$  be the ideal in  $T(\mathfrak{g})$  that is generated by the elements of the form

$$X \otimes Y - Y \otimes X. \quad (4.1.10)$$

Then  $T(\mathfrak{g})/I$  is the symmetric algebra  $S(\mathfrak{g})$ . On the other hand, let  $J$  be the ideal in  $T(\mathfrak{g})$  generated by the elements of the form

$$X \otimes Y - Y \otimes X - [X, Y]. \quad (4.1.11)$$

Then  $T(\mathfrak{g})/J$  is the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ . It is customary to denote  $X_1 \otimes \cdots \otimes X_n + I$  in  $S(\mathfrak{g})$  or  $X_1 \otimes \cdots \otimes X_n + J$  in  $\mathcal{U}(\mathfrak{g})$  simply as  $X_1 \cdots X_n$ . In any case, the elements  $X_1 \cdots X_n$ , where  $X_i$  ( $i \in \{1, \dots, n\}$ ,  $n \in \mathbb{N}$ ) are vectors in  $\mathfrak{g}$ , generate  $S(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{g})$  as vector spaces.

Consider the symmetrization map

$$\begin{aligned} \text{PBW} : S(\mathfrak{g}) &\rightarrow \mathcal{U}(\mathfrak{g}), \\ X_1 \cdots X_n &\mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}. \end{aligned} \quad (4.1.12)$$

Here  $S_n$  denotes the symmetric group of degree  $n$ . It was demonstrated by H. Poincaré [80], G. Birkhoff [14], and E. Witt [100] that the above map is a *vector space* isomorphism; hence it is called the *Poincaré-Birkhoff-Witt isomorphism*.

Here we review a proof for the Poincaré-Birkhoff-Witt isomorphism using distribution theory. Start by considering the algebra isomorphism 4.1.8 for  $A = \mathfrak{g}$  and  $A = G$ , respectively:

$$\begin{aligned} \delta_{\mathfrak{g}} : S(\mathfrak{g}) &\xrightarrow{\sim} \mathcal{E}'_0(\mathfrak{g}), \\ \delta_G : \mathcal{U}(\mathfrak{g}) &\xrightarrow{\sim} \mathcal{E}'_e(G). \end{aligned}$$

Here  $\mathcal{E}'_0(\mathfrak{g})$  is the algebra of distributions on  $\mathfrak{g}$  with support  $\{0\}$ , and  $\mathcal{E}'_e(G)$  is the algebra of distributions on  $G$  with support  $\{e\}$ . Next, consider the exponential map  $\exp : \mathfrak{g} \rightarrow G$ , which is a local diffeomorphism near 0. Since functions pull-back along smooth maps of manifolds, their dual objects, the distributions, push-forward. More precisely, the pushforward of  $\Lambda$  in  $\mathcal{E}'_0(\mathfrak{g})$  along the exponential map is the distribution  $\exp_* \Lambda$  in  $\mathcal{E}'_e(G)$  defined by

$$\langle \exp_* \Lambda, f \rangle := \langle \Lambda, \exp^* f \rangle$$

for smooth functions  $f$  on  $G$ . Because the exponential map is a local diffeomorphism, the push-forward map

$$\exp_* : \mathcal{E}'_0(\mathfrak{g}) \rightarrow \mathcal{E}'_e(G) \quad (4.1.13)$$

is a vector space isomorphism. Then, demonstrating that the following diagram is commutative yields a proof for the Poincaré-Birkhoff-

Witt isomorphism:

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\text{PBW}} & \mathcal{U}(\mathfrak{g}) \\ \delta_{\mathfrak{g}} \downarrow \wr & & \wr \downarrow \delta_G \\ \mathcal{E}'_0(\mathfrak{g}) & \xrightarrow[\exp_*]{\sim} & \mathcal{E}'_e(G) \end{array} \quad (4.1.14)$$

Let us verify this beginning with the simple case. Let  $X$  be a vector in  $\mathfrak{g}$ , and view it as a generator in  $S(\mathfrak{g})$ . Take its  $k$ th power  $X^k$  in  $S(\mathfrak{g})$ . We wish to check that

$$\exp_*(\delta_{\mathfrak{g}}(X^k)) = \delta_G(\text{PBW}(X^k)). \quad (4.1.15)$$

Since  $\delta_{\mathfrak{g}}$  and  $\delta_G$  are multiplicative and  $\text{PBW}(X^k) = X^k$ , we may just show that

$$\exp_*(\delta_{\mathfrak{g}}(X)^k) = \delta_G(X)^k. \quad (4.1.16)$$

Let  $f$  be a smooth function on  $G$ . By the definition of the push-forward map  $\exp_*$ ,

$$\langle \exp_*(\delta_{\mathfrak{g}}(X)^k), f \rangle = \langle \delta_{\mathfrak{g}}(X)^k, \exp^* f \rangle. \quad (4.1.17)$$

By the definition of the convolution product of distributions, we have, for the right-hand side of Equation 4.1.17,

$$\langle \delta_{\mathfrak{g}}(X)^k, \exp^* f \rangle = \langle \delta_{\mathfrak{g}}(X) \otimes \cdots \otimes \delta_{\mathfrak{g}}(X), \widehat{\exp^* f} \rangle, \quad (4.1.18)$$

where  $\widehat{\exp^* f}$  is a smooth function  $\mathfrak{g}^k$  defined by

$$\widehat{\exp^* f}(X_1, \dots, X_k) = \exp^* f(X_1 + \cdots + X_k),$$

and the  $i$ th factor of  $\delta_{\mathfrak{g}}(X) \otimes \cdots \otimes \delta_{\mathfrak{g}}(X)$  pairs with  $\widehat{\exp^* f}$  by viewing it as a function that depends only on the  $i$ th variable. Applying Equation 4.1.7 iteratively, we get

$$\begin{aligned} & \langle \delta_{\mathfrak{g}}(X) \otimes \cdots \otimes \delta_{\mathfrak{g}}(X), \widehat{\exp^* f} \rangle \\ &= \frac{d}{dt_1} \Big|_{t_1=0} \cdots \frac{d}{dt_k} \Big|_{t_k=0} \exp^* f(t_1 X + \cdots + t_k X) \\ &= \frac{d}{dt_1} \Big|_{t_1=0} \cdots \frac{d}{dt_k} \Big|_{t_k=0} f(\exp(t_1 X + \cdots + t_k X)) \\ &= \frac{d}{dt_1} \Big|_{t_1=0} \cdots \frac{d}{dt_k} \Big|_{t_k=0} f(\exp(t_1 X) \cdots \exp(t_k X)) \\ &= (\tilde{X} \cdots \tilde{X})_e f = \langle \delta_G(\tilde{X} \cdots \tilde{X}), f \rangle \\ &= \langle \delta_G(\tilde{X}) * \cdots * \delta_G(\tilde{X}), f \rangle. \end{aligned} \quad (4.1.19)$$

Equations 4.1.17–4.1.19 prove Equation 4.1.16, which is equivalent to Equation 4.1.15. For the general case, just put  $X = t_1 X_1 + \cdots + t_n X_n$  into Equation 4.1.15, where  $t_1, \dots, t_n$  are indeterminates, and compare both sides of the resulting equation.

4.1.20 Let  $C_c^\infty(M)$  be the space of compactly supported smooth functions on a manifold  $M$ . Suppose the Lie group  $A$  acts on  $M$  on the left. This induces a  $G$ -action on  $C_c^\infty(M)$  by Equation 4.1.2. Since distributions are dual to functions, we define the  $G$ -action on  $\mathcal{D}'(M)$  in the same manner as in Equation 2.2.7; that is, for  $g \in A$  and  $\varphi \in \mathcal{D}'(M)$ , we define the distribution  $g \cdot \varphi$  on  $M$  by setting its pairing with  $f \in C_c^\infty(M)$  as

$$\langle g \cdot \varphi, f \rangle := \langle \varphi, g^{-1} \cdot f \rangle.$$

Now consider the  $G$ -action on  $\mathfrak{g}$  by the adjoint representation, and on  $G$  by conjugation. These induce  $G$ -actions on  $\mathcal{E}'_0(\mathfrak{g})$  and  $\mathcal{E}'_e(G)$  as explained above; we shall denote their  $G$ -invariant parts, respectively, by  $\mathcal{E}'_0(\mathfrak{g})^G$  and  $\mathcal{E}'_e(G)^G$ . The linear isomorphism  $\exp_*$  then maps  $\mathcal{E}'_0(\mathfrak{g})^G$  onto  $\mathcal{E}'_e(G)^G$ . (This owes to Equation 2.2.6.) Meanwhile, the preimage of  $\mathcal{E}'_0(\mathfrak{g})^G$  under the map  $\delta_{\mathfrak{g}}$  is  $S(\mathfrak{g})^{\mathfrak{g}}$ . Likewise, the preimage of  $\mathcal{E}'_e(G)^G$  under  $\delta_G$  is  $\mathcal{Z}(\mathfrak{g})$ . So the commutative diagram 4.1.14, restricted to the invariant parts, gives us

$$\begin{array}{ccc} S(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{\text{PBW}} & \mathcal{Z}(\mathfrak{g}) \\ \delta_{\mathfrak{g}} \downarrow \wr & & \wr \downarrow \delta_G \\ \mathcal{E}'_0(\mathfrak{g})^G & \xrightarrow[\exp_*]{} & \mathcal{E}'_e(G)^G \end{array} \quad (4.1.21)$$

4.1.22 DUFLO ISOMORPHISM. The reason we gave an interpretation of the Poincaré-Birkhoff-Will isomorphism in the language of distribution is that the Duflo isomorphism is simpler to state in terms of distributions rather than that of differential operators. In short, the Duflo isomorphism is a modification of the restricted Poincaré-Birkhoff-Witt isomorphism  $\text{PBW} : S(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathcal{Z}(\mathfrak{g})$  so that it becomes an *algebra* isomorphism. The corresponding modification of  $\exp_*$  is (see the diagram 4.1.21)

$$\exp_* \circ j \quad (4.1.23)$$

where  $j$  denotes the multiplication by the function

$$j(X) = \det^{1/2} \left( \frac{\sinh \text{ad}_X / 2}{\text{ad}_X / 2} \right). \quad (4.1.24)$$

Thus, the Duflo isomorphism is the algebra isomorphism

$$\text{Duf} = S(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathcal{Z}(\mathfrak{g})$$

defined by

$$\text{Duf} = (\delta_G)^{-1} \circ (\exp_* \circ j) \circ \delta_{\mathfrak{g}}. \quad (4.1.25)$$

When  $\mathfrak{g}$  is a complex semisimple Lie algebra, the Duflo isomor-



phism is (the inverse of) the Harish-Chandra isomorphism<sup>1</sup>. Duflo [32] generalized the isomorphism  $S(\mathfrak{g})^{\mathfrak{g}} \simeq \mathcal{Z}(\mathfrak{g})$  for all finite-dimensional Lie algebra and proved it using Kirillov's orbit method. Alekseev and Meinrenken [2] gave an algebraic proof for the Duflo isomorphism for quadratic Lie algebras. A sketch of their proof and the algebraic description of the Duflo isomorphism is presented in Section 5.3.19.

**4.1.26  $S(\mathfrak{g})$  AS THE ASSOCIATED GRADED ALGEBRA OF  $\mathcal{U}(\mathfrak{g})$ .** Let  $I$  and  $J$  be the two ideals in  $T(\mathfrak{g})$  that we defined in Section 4.1.9 so that  $T(\mathfrak{g})/I = S(\mathfrak{g})$  and  $T(\mathfrak{g})/J = \mathcal{U}(\mathfrak{g})$ . Imposing the relation defined by  $I$  on an element in the homogeneous component  $T^k(\mathfrak{g})$  of  $T(\mathfrak{g})$  of degree  $k$  yields again an element in  $T^k(\mathfrak{g})$ , whereas the relation defined by  $J$  could yield an element in  $T^\ell(\mathfrak{g})$  with  $\ell < k$ . Thus, the grading on the tensor algebra  $T(\mathfrak{g})$  induces a graded algebra structure on the symmetric algebra:

$$S(\mathfrak{g}) = \bigoplus_{k \in \mathbb{Z}} S^k(\mathfrak{g}),$$

where  $S^k(\mathfrak{g}) = T^k(\mathfrak{g})/I$ . But the same grading on  $T(\mathfrak{g})$  induces a filtered algebra structure on the universal enveloping algebra:

$$\mathcal{U}(\mathfrak{g}) = \bigcup_{k \in \mathbb{Z}} \mathcal{U}_k(\mathfrak{g}),$$

where  $\mathcal{U}_k(\mathfrak{g}) = (\bigoplus_{\ell=0}^k T^\ell(\mathfrak{g}))/J$ .

For any filtered algebra  $A = \bigcup_{n \in \mathbb{Z}} A_n$  (over a field), its *associated graded algebra* is defined as

$$\text{gr } A = \bigoplus_{n \in \mathbb{Z}} A_n/A_{n-1},$$

where the product structure is defined by

$$\begin{aligned} A_k/A_{k-1} \times A_\ell/A_{\ell-1} &\rightarrow A_{k+\ell}/A_{k+\ell-1} \\ (x + A_{k-1}, y + A_{\ell-1}) &\mapsto xy + A_{k+\ell-1}. \end{aligned}$$

The canonical projection

$$\sigma_k : A_k \rightarrow A_k/A_{k-1}$$

is called the *symbol map of order  $k$* . The maps  $\sigma_k$ ,  $k \in \mathbb{Z}$ , collectively

<sup>1</sup> The Harish-Chandra isomorphism [50, Lem. 36, Lem. 38] for a complex semisimple Lie algebra  $\mathfrak{g}$  is usually presented as an algebra isomorphism  $\gamma : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W$  where  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$  (which means that  $\mathfrak{h}$  is a maximal abelian subalgebra of  $\mathfrak{g}$  such that  $\text{ad}(\mathfrak{h})$  is simultaneously diagonalizable) and  $S(\mathfrak{h})^W$  is the subalgebra of  $S(\mathfrak{h})$  consisting of elements that are invariant under the action of the Weyl group (the group generated by the reflections with respect to the root hyperplanes like the ones we have seen for the Lie algebra of a maximal torus in a compact Lie group). The restriction of a polynomial function on  $\mathfrak{g}^*$  to  $\mathfrak{h}^*$  induces a surjective map  $S(\mathfrak{g}) \rightarrow S(\mathfrak{h})$  whose restriction to the  $\text{ad}(\mathfrak{g})$ -invariants yields an algebra isomorphism  $\pi : S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^W$ . So  $\pi^{-1} \circ \gamma$  yields the algebra isomorphism  $\mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{g})^{\mathfrak{g}}$ , to which the Duflo isomorphism is the inverse. For more on the Harish-Chandra isomorphism, see [59, § 23.3] or [62, Thm. 5.44].

define the *symbol map*

$$\sigma : A \rightarrow \text{gr } A = \bigoplus_{k \in \mathbb{Z}} A_k / A_{k-1}.$$

(The above definitions appear in [27, 85].)

The point we are trying to make is that the Poincaré-Birkhoff-Witt isomorphism implies that  $S(\mathfrak{g})$  is isomorphic to  $\text{gr } \mathcal{U}(\mathfrak{g})$ . To see this, choose a basis  $\{X_i\}_{i=1}^n$  for  $\mathfrak{g}$ . Then the following set of simple tensors,

$$B_k := \{X_{i_1} \otimes \cdots \otimes X_{i_k} \in T^k(\mathfrak{g}) \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n\},$$

yields, under the quotient map  $T(\mathfrak{g}) \rightarrow T(\mathfrak{g})/I$ , a basis for the homogeneous component of  $S(\mathfrak{g})$  of degree  $k$ ; that is, we have a vector space isomorphism

$$\begin{aligned} \text{Span}_{\mathbb{R}} B_k &\rightarrow S^k(\mathfrak{g}), \\ X_{i_1} \otimes \cdots \otimes X_{i_k} &\mapsto X_{i_1} \cdots X_{i_k}. \end{aligned}$$

Here  $\text{Span}_{\mathbb{R}} B_k$  denotes the  $\mathbb{R}$ -linear span of  $B_k$ . Meanwhile, the same simple tensors  $X_{i_1} \otimes \cdots \otimes X_{i_k}$  can be used to define the following linear map:

$$\begin{aligned} \text{Span}_{\mathbb{R}} B_k &\rightarrow \mathcal{U}_k(\mathfrak{g}) / \mathcal{U}_{k-1}(\mathfrak{g}), \\ X_{i_1} \otimes \cdots \otimes X_{i_k} &\mapsto X_{i_1} \cdots X_{i_k} + \mathcal{U}_{k-1}(\mathfrak{g}). \end{aligned} \quad (4.1.27)$$

This map is clearly surjective. To see that it is injective, note that it factors through the quotient map  $\pi_J : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})/J$ . Now permuting the vectors that constitute the simple tensor  $X_{i_1} \otimes \cdots \otimes X_{i_k}$  in  $B_k$  does not give another simple tensor in  $B_k$ . As a consequence, the relations in  $\mathcal{U}(\mathfrak{g})$  defined by the ideal  $J$  cannot be applied to a nonzero linear combination of elements in  $\pi_J(B_k)$  to obtain an element of  $\mathcal{U}_{k-1}(\mathfrak{g})$ . Therefore, the map 4.1.27 is injective. Hence, we have a linear isomorphism:

$$\begin{aligned} \bigoplus_{k \in \mathbb{Z}} S^k(\mathfrak{g}) &\rightarrow \bigoplus_{k \in \mathbb{Z}} \mathcal{U}_k(\mathfrak{g}) / \mathcal{U}_{k-1}(\mathfrak{g}), \\ X_{i_1} \cdots X_{i_k} &\mapsto X_{i_1} \cdots X_{i_k} + \mathcal{U}_{k-1}(\mathfrak{g}). \end{aligned} \quad (4.1.28)$$

To prove that this map is multiplicative, we must show that

$$\begin{aligned} (X_{i_1} \cdots X_{i_k})(X_{j_1} \cdots X_{j_\ell}) + \mathcal{U}_{k+\ell-1}(\mathfrak{g}) \\ = (X_{j_1} \cdots X_{j_\ell})(X_{i_1} \cdots X_{i_k}) + \mathcal{U}_{k+\ell-1}(\mathfrak{g}). \end{aligned}$$

This would be true if  $\text{gr } \mathcal{U}(\mathfrak{g})$  is commutative, which is the case since the commutator of two generators  $X$  and  $Y$  in  $\mathfrak{g}$  is zero, that is,

$$XY = YX \pmod{\mathcal{U}_1(\mathfrak{g})}.$$

To sum up,  $S(\mathfrak{g})$  and  $\text{gr } \mathcal{U}(\mathfrak{g})$  are isomorphic as algebras..

Notice that the isomorphism 4.1.28 is equal to the composition  $\sigma \circ \text{PBW}$ , where  $\sigma : \mathcal{U}(\mathfrak{g}) \rightarrow \text{gr } \mathcal{U}(\mathfrak{g})$  is the symbol map. So we have a

commutative diagram:

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\text{PBW}} & \mathcal{U}(\mathfrak{g}) \\ (4.1.28) \downarrow \wr & \swarrow \sigma & \\ \text{gr } \mathcal{U}(\mathfrak{g}) & & \end{array}$$

Thus, under the identification of  $S(\mathfrak{g})$  as the associated graded algebra of  $\mathcal{U}(\mathfrak{g})$ , we may say that the Poincaré-Birkhoff-Witt isomorphism is the inverse of the symbol map.

## 4.2 THE ASYMPTOTIC EXPANSION OF THE HEAT KERNEL

4.2.1 The perspective we took in the previous section was that of distribution theory. We now return to the differential operator point of view.

4.2.2 DEFINITION. Let  $D(\mathfrak{g})$  be the algebra of constant coefficient differential operators on  $\mathfrak{g}$ . Let  $D(G)$  be the algebra of left-invariant differential operators on  $G$ . Consider the  $G$ -action on  $\mathfrak{g}$  by the adjoint representation, and on  $G$  by conjugation; this induces  $G$ -actions on  $D(\mathfrak{g})$  and  $D(G)$  (see Section 4.1.1); denote the  $G$ -invariant parts by  $D(\mathfrak{g})^G$  and  $D(G)^G$ , respectively. (Then  $D(G)^G$  is the algebra of bi-invariant differential operators on  $G$  as before.) Owing to the algebra isomorphism 4.1.4, we can identify  $D(\mathfrak{g})^G$  with  $S(\mathfrak{g})^G$  and  $D(G)^G$  with  $\mathcal{Z}(G)$ . The Duflo isomorphism  $\text{Duf} : S(\mathfrak{g})^G \xrightarrow{\sim} \mathcal{Z}(G)$  then induces the algebra isomorphism

$$\overline{\text{Duf}} : D(\mathfrak{g})^G \xrightarrow{\sim} D(G)^G$$

so that we have the following commutative diagram:

$$\begin{array}{ccc} S(\mathfrak{g})^G & \xrightarrow{\text{Duf}} & \mathcal{Z}(G) \\ \tau_{\mathfrak{g}} \downarrow & & \downarrow \tau_G \\ D(\mathfrak{g})^G & \xrightarrow{\overline{\text{Duf}}} & D(G)^G \\ \varepsilon_{\mathfrak{g}} \downarrow & & \downarrow \varepsilon_G \\ \mathcal{E}'_0(\mathfrak{g})^G & \xrightarrow{\exp_* \circ j} & \mathcal{E}'_e(G)^G \end{array} \quad \begin{array}{c} \delta_{\mathfrak{g}} \quad \delta_G \end{array} \quad (4.2.3)$$

Here the maps  $\tau_{\mathfrak{g}}$  and  $\tau_G$  are provided by (the appropriate restrictions of) the isomorphism 4.1.4 for  $A = \mathfrak{g}$  and  $A = G$ , respectively. Likewise, the maps  $\varepsilon_{\mathfrak{g}}$  and  $\varepsilon_G$  are provided by the isomorphism 4.1.5. The arrows in the above diagram are all algebra isomorphisms.

4.2.4 PROPOSITION. Let  $\Delta_{\mathfrak{g}}$  and  $\Delta_G$  be the Laplacians on  $\mathfrak{g}$  and  $G$ , respectively. Then

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}) = \Delta_G - \|\rho\|^2. \quad (4.2.5)$$

Here  $\rho$  is half the sum of the positive roots of  $G$ , and  $\|\cdot\|$  is the norm induced by the bi-invariant metric on  $G$ .

*Proof.* By the definition of the map  $\overline{\text{Duf}}$ ,

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}) = \tau_G \circ \text{Duf} \circ \tau_g^{-1}(\Delta_{\mathfrak{g}}). \quad (4.2.6)$$

Let  $\{X_1, \dots, X_n\}$  be any orthonormal basis for  $\mathfrak{g}$ . The map  $\tau_g$  maps  $X_i$  to the partial derivative  $\partial_i$  on  $\mathfrak{g}$  with respect to the vector  $X_i$ . So  $\tau_g(\sum_{i=1}^n X_i X_i) = \sum_{i=1}^n \partial_i^2$ , which is  $\Delta_{\mathfrak{g}}$ . Therefore,

$$\tau_g^{-1}(\Delta_{\mathfrak{g}}) = \sum_{i=1}^n X_i X_i. \quad (4.2.7)$$

It is known that the image of this element under the Duflo isomorphism satisfies

$$\text{Duf}\left(\sum_{i=1}^n X_i X_i\right) = \Omega + \frac{1}{24} \text{tr}_g(\Omega), \quad (4.2.8)$$

where  $\Omega$  is the Casimir element in  $\mathcal{Z}(\mathfrak{g})$  and  $\text{tr}_g(\Omega)$  is the trace of the Casimir for its action on  $\mathfrak{g}$  arising from the adjoint representation  $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ . An algebraic proof of Equation 4.2.8 can be found in the work of A. Alekseev and E. Meinrenken [3]. We shall review their theory in Chapter 5. In particular it is Equation 5.4.23 with  $\mathfrak{g} = \mathfrak{k}$  that gives Equation 4.2.8.

Owing to a result of B. Kostant [65, Eq. 1.85],

$$\frac{1}{24} \text{tr}_g \Omega = -\|\rho\|^2. \quad (4.2.9)$$

Inserting this into Equation 4.2.8,

$$\text{Duf}\left(\sum_{i=1}^n X_i X_i\right) = \Omega - \|\rho\|^2. \quad (4.2.10)$$

So, by Equations 4.2.6, 4.2.7 and 4.2.10, we have

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}) = \tau_G(\Omega - \|\rho\|^2) = \tau_G(\Omega) - \|\rho\|^2.$$

Since  $\tau_G(\Omega) = \Delta_G$  (Equation 2.2.18), we have

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}) = \Delta_G - \|\rho\|^2. \quad \square$$

**4.2.11 DEFINITION.** Recall that the exponential map is a local diffeomorphism near 0 in  $\mathfrak{g}$ , that is, there is an open neighborhood  $V$  of 0 that is mapped diffeomorphically onto  $\exp(V) =: U$  under the exponential map. Denote the diffeomorphism of  $V$  onto  $U$  via the exponential map by

$$\exp_V : V \rightarrow U. \quad (4.2.12)$$

For an invariant differential operator  $D$  on  $G$ , let  $D^{\text{exp}}$  be the constant coefficient differential operator on  $V$  such that,

$$D^{\text{exp}}\phi = \exp_V^*(D(\exp_V^{-1})^*\phi).$$

Since  $D^{\text{exp}}$  is a constant coefficient differential operator, it makes sense to extend it to a constant coefficient differential operator on  $\mathfrak{g}$ .

So we have a linear map

$$\begin{aligned} D(G) &\rightarrow D(\mathfrak{g}), \\ D &\mapsto D^{\exp}. \end{aligned} \quad (4.2.13)$$

*Remark.* (1) Suppose  $f$  is a smooth function on  $G$ . Let  $f_U$  be its restriction to  $U$ , and set

$$f^{\exp} := \exp_V^* f_U. \quad (4.2.14)$$

Then,

$$D^{\exp} f^{\exp} = (Df)^{\exp}. \quad (4.2.15)$$

In a word,  $f^{\exp}$  and  $D^{\exp}$  are the expressions for  $f$  and  $D$  under a local exponential chart near the identity in  $G$ .

- (2) Owing to the algebra isomorphism 4.1.5, we can identify  $D(\mathfrak{g})$  with  $\mathcal{E}'_0(\mathfrak{g})$  and  $D(G)$  with  $\mathcal{E}'_e(G)$ . Then the inverse of the vector space isomorphism 4.1.13 induces a linear isomorphism

$$\widetilde{\exp}_*^{-1} : D(G) \xrightarrow{\sim} D(\mathfrak{g}) \quad (4.2.16)$$

so that we have the following commutative diagram:

$$\begin{array}{ccc} D(\mathfrak{g}) & \xleftarrow{\widetilde{\exp}_*^{-1}} & D(G) \\ \varepsilon_{\mathfrak{g}} \downarrow \wr & & \wr \downarrow \varepsilon_G \\ \mathcal{E}'_0(\mathfrak{g}) & \xleftarrow{\exp_*^{-1}} & \mathcal{E}'_e(G) \end{array} \quad (4.2.17)$$

The linear isomorphism  $\widetilde{\exp}_*^{-1}$  is precisely the map 4.2.13;

$$\widetilde{\exp}_*^{-1}(D) = D^{\exp}. \quad (4.2.18)$$

- (3) Consider the  $G$ -action on  $\mathfrak{g}$  by the adjoint representation, and on  $G$  by conjugation. Then the commutative diagram 4.2.17, restricted to the invariant parts, yields

$$\begin{array}{ccc} D(\mathfrak{g})^G & \xleftarrow{\widetilde{\exp}_*^{-1}} & D(G)^G \\ \varepsilon_{\mathfrak{g}} \downarrow \wr & & \wr \downarrow \varepsilon_G \\ \mathcal{E}'_0(\mathfrak{g})^G & \xleftarrow{\exp_*^{-1}} & \mathcal{E}'_e(G)^G \end{array} \quad (4.2.19)$$

**4.2.20 PROPOSITION.** Let  $j(X) = \det^{1/2}(\frac{\sinh \text{ad}_X/2}{\text{ad}_X/2})$ . We have

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\exp} = j^{-1} \circ \Delta_{\mathfrak{g}} \circ j.$$

*Remark.* The above equation is an equality of differential operators;  $j$  and  $j^{-1}$  are to be read as the multiplication operators by the respective functions.

*Proof.* By Equation 4.2.18,

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} = \widetilde{\text{exp}}_*^{-1}(\overline{\text{Duf}}(\Delta_{\mathfrak{g}})).$$

Then, by the commutative diagram 4.2.19,

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} = \varepsilon_{\mathfrak{g}}^{-1} \circ \text{exp}_*^{-1} \circ \varepsilon_{\mathfrak{g}}(\overline{\text{Duf}}(\Delta_{\mathfrak{g}})). \quad (4.2.21)$$

Now, by the commutative diagram 4.2.3,

$$\begin{aligned} \varepsilon_{\mathfrak{g}}(\overline{\text{Duf}}(\Delta_{\mathfrak{g}})) &= \text{exp}_* \circ j \circ \varepsilon_{\mathfrak{g}}(\Delta_{\mathfrak{g}}) = \text{exp}_* \circ j(\Delta_{\mathfrak{g},0}) \\ &= \text{exp}_*((\Delta_{\mathfrak{g}} \circ j)_0). \end{aligned} \quad (4.2.22)$$

Inserting this into Equation 4.2.21, we get

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} = \varepsilon_{\mathfrak{g}}^{-1}((\Delta_{\mathfrak{g}} \circ j)_0).$$

This implies that

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} = \phi \circ \Delta_{\mathfrak{g}} \circ j \quad (4.2.23)$$

for some smooth function  $\phi$  on  $\mathfrak{g}$  such that  $\phi(0) = 1$ .

It remains to show that  $\phi = 1/j$ . Since  $\overline{\text{Duf}}$  is an algebra isomorphism, we have

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}^2) = (\overline{\text{Duf}} \Delta_{\mathfrak{g}})^2.$$

Hence,

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}^2)^{\text{exp}} = \overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} \circ \overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}}. \quad (4.2.24)$$

Repeating the argument we gave for Equation 4.2.23 word for word, except replacing  $\Delta_{\mathfrak{g}}$  with  $\Delta_{\mathfrak{g}}^2$ , we get

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}}^2)^{\text{exp}} = \phi \circ \Delta_{\mathfrak{g}}^2 \circ j. \quad (4.2.25)$$

Meanwhile, by Equation 4.2.23,

$$\overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} \circ \overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}} = \phi \circ \Delta_{\mathfrak{g}} \circ (j\phi) \circ \Delta_{\mathfrak{g}} \circ j. \quad (4.2.26)$$

So, by Equations 4.2.24–4.2.26,

$$\Delta_{\mathfrak{g}}^2 = \Delta_{\mathfrak{g}} \circ \psi \circ \Delta_{\mathfrak{g}} \quad (4.2.27)$$

where  $\psi := j\phi$ . We need to show that  $\psi \equiv 1$ . Since  $\psi(0) = 1$ , it is sufficient to show that  $\text{grad } \psi \equiv 0$ , where  $\text{grad}$  is the usual gradient operator on the smooth functions on the Euclidean space  $\mathfrak{g}$ . For an arbitrary vector  $Y$  in  $\mathfrak{g}$ , define the smooth functions  $s_Y$  and  $c_Y$  on  $\mathfrak{g}$  by  $s_Y(X) = \sin\langle Y, X \rangle$  and  $c_Y(X) = \cos\langle Y, X \rangle$ . Then,

$$\Delta_{\mathfrak{g}}(s_Y) = -\|Y\|^2 s_Y.$$

So

$$\Delta_{\mathfrak{g}}^2(s_Y) = \|Y\|^4 s_Y. \quad (4.2.28)$$

Whereas,

$$\begin{aligned}
& \Delta_{\mathfrak{g}} \circ \psi \circ \Delta_{\mathfrak{g}}(s_Y) \\
&= -\|Y\|^2 \Delta_{\mathfrak{g}}(\psi s_Y) \\
&= -\|Y\|^2 (\Delta_{\mathfrak{g}}(\psi) s_Y + 2\langle \text{grad } \psi, \text{grad } s_Y \rangle + \psi \Delta_{\mathfrak{g}}^2(s_Y)) \\
&= -\|Y\|^2 (\Delta_{\mathfrak{g}}(\psi) - \|Y\|^2 \psi) s_Y + 2\langle \text{grad } \psi, Y \rangle c_Y. \tag{4.2.29}
\end{aligned}$$

Hence, by Equations 4.2.27–4.2.29, we have

$$\|Y\|^2 (\Delta_{\mathfrak{g}}(\psi) - \|Y\|^2 \psi + \|Y\|^2) s_Y = 2\langle \text{grad } \psi, Y \rangle c_Y.$$

Since  $s_Y$  and  $c_Y$  are linearly independent, we have  $\langle \text{grad } \psi, Y \rangle = 0$ . Because  $Y$  is arbitrary, we have  $\text{grad } \psi \equiv 0$  as desired.  $\square$

Combining Propositions 4.2.4 and 4.2.20 gives us the following known result (see [55, Thm. 3.15, p. 273]):

**4.2.30 COROLLARY.** *The Laplacian  $\Delta_G$  on  $G$ , under the local exponential chart near the identity, takes the following form:*

$$\Delta_G^{\text{exp}} = j^{-1} \circ \Delta_{\mathfrak{g}} \circ j + \|\rho\|^2.$$

**4.2.31 PROPOSITION.** *Let  $p_t$  be the heat convolution kernel of  $\text{Duf}(\Delta_{\mathfrak{g}})$ . (For the definition of the heat convolution kernel, see Section 2.1.9.) Then  $p_t^{\text{exp}} := \exp^* p_t$  has the asymptotic expansion*

$$p_t^{\text{exp}} \sim h_t \frac{1}{j}$$

for  $t \rightarrow 0+$ , valid in some neighborhood of  $0 \in \mathfrak{g}$ , where  $h_t$  is the Gaussian kernel on  $\mathfrak{g}$ , that is,  $h_t(X) = e^{-\|X\|/4t} (4\pi t)^{-\dim(\mathfrak{g})/2}$ . In other words, the coefficients of the asymptotic expansion 2.1.15, for this case, are  $a_0(x) = 1/j(\log(x))$  and  $a_n(x) = 0$  for  $n \in \mathbb{N}$ .

*Remark.* The reason  $h_t/j$  is not the exact heat kernel is that the exponential map fails to be a global diffeomorphism.

*Proof.* Writing the heat equation,  $(\partial_t + \overline{\text{Duf}}(\Delta_{\mathfrak{g}}))p_t = 0$ , in the exponential chart near the identity in  $G$ , we get

$$(\partial_t + \overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}})p_t^{\text{exp}} = 0.$$

Let  $s_t := h_t \sum_{i=0}^{\infty} a_i t^i$  be the asymptotic expansion for  $p_t^{\text{exp}}$ . It is the formal solution to

$$(\partial_t + \overline{\text{Duf}}(\Delta_{\mathfrak{g}})^{\text{exp}})s_t = 0. \tag{4.2.32}$$

By Proposition 4.2.20, the differential equation 4.2.32 is equivalent to

$$\frac{1}{j}(\partial_t - \Delta_{\mathfrak{g}})j s_t = 0.$$

We need to show that  $s_t = h_t/j$  satisfies this differential equation. But that is easily seen from the fact that  $h_t$  satisfies the heat equation  $(\partial_t - \Delta_{\mathfrak{g}})h_t = 0$ .  $\square$

4.2.33 LEMMA. Let  $G$  be a compact connected Lie group equipped with a bi-invariant metric  $\langle \cdot, \cdot \rangle$ . The value of the scalar curvature  $S$  of  $G$  is given by

$$S = -\frac{1}{4} \operatorname{tr}_{\mathfrak{g}}(\Omega) = 6\|\rho\|^2, \quad (4.2.34)$$

where  $\rho$  is half the sum of the positive roots of  $G$ .

*Proof.* As we pointed out in Section 2.2.14, the Riemannian connection  $\nabla$  on  $G$  satisfies  $\nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2}[\tilde{X}, \tilde{Y}]$ . So the curvature tensor  $R$  satisfies

$$\begin{aligned} R(\tilde{X}, \tilde{Y})\tilde{Z} &= \nabla_{\tilde{X}} \nabla_{\tilde{Y}} \tilde{Z} - \nabla_{\tilde{Y}} \nabla_{\tilde{X}} \tilde{Z} - \nabla_{[\tilde{X}, \tilde{Y}]} \tilde{Z} && \text{(by definition)} \\ &= \frac{1}{4}[\tilde{X}, [\tilde{Y}, \tilde{Z}]] - \frac{1}{4}[\tilde{Y}, [\tilde{X}, \tilde{Z}]] - \frac{1}{2}[[\tilde{X}, \tilde{Y}], \tilde{Z}]. \end{aligned}$$

Applying the Jacobi identity to the above, we get

$$R(\tilde{X}, \tilde{Y})\tilde{Z} = -\frac{1}{4}[[\tilde{X}, \tilde{Y}], \tilde{Z}].$$

Then, the Riemann curvature tensor  $Rm$  satisfies

$$\begin{aligned} Rm(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= \langle R(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W} \rangle && \text{(by definition)} \\ &= -\frac{1}{4} \langle [[\tilde{X}, \tilde{Y}], \tilde{Z}], \tilde{W} \rangle \\ &= -\frac{1}{4} \langle [[X, Y], Z], W \rangle, \end{aligned} \quad (4.2.35)$$

where the last equality owes to the left-invariance of the metric. Now the scalar curvature  $S$  is, by definition, the trace of the Ricci tensor; this means, in terms of an orthonormal basis  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  for  $\mathfrak{g}$ ,

$$S = \sum_{i,j=1}^{\dim \mathfrak{g}} Rm(\tilde{X}_i, \tilde{X}_j, \tilde{X}_j, \tilde{X}_i). \quad (4.2.36)$$

So, by Equation 4.2.35,

$$\begin{aligned} S &= -\frac{1}{4} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle [[X_i, X_j], X_j], X_i \rangle \\ &= -\frac{1}{4} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle [\operatorname{ad}_{X_j} \circ \operatorname{ad}_{X_j}(X_i), X_i] \rangle \\ &= -\frac{1}{4} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle \operatorname{ad}(\Omega)X_i, X_i \rangle = -\frac{1}{4} \operatorname{tr}_{\mathfrak{g}}(\Omega). \end{aligned}$$

This proves the first equality in Equation 4.2.34. The second equality follows from Equation 4.2.9.  $\square$

4.2.37 THEOREM. Let  $G$  be a compact connected Lie group equipped with a bi-invariant metric. Let  $k_t$  be the heat convolution kernel of the



Laplacian on  $G$ . Then  $k_t^{\text{exp}} := \exp^* k_t$  has the asymptotic expansion

$$k_t^{\text{exp}} \sim h_t \frac{e^{tS/6}}{j} = h_t \sum_{n=0}^{\infty} \frac{1}{j} \frac{1}{n!} \left(\frac{S}{6}\right)^n t^n \quad (4.2.38)$$

for  $t \rightarrow 0+$ , valid in a neighborhood of  $0 \in \mathfrak{g}$ . Here  $j$  is the function defined by the power series 4.1.24,  $S$  is the scalar curvature of  $G$ , and  $h_t$  is the Gaussian kernel on  $\mathfrak{g}$ , that is,  $h_t(X) = e^{-\|X\|^2/4t} (4\pi t)^{-\dim(\mathfrak{g})/2}$ .

*Proof.* By Proposition 4.2.4, we have

$$\Delta_G = \overline{\text{Duf}}(\Delta_{\mathfrak{g}}) + \|\rho\|^2.$$

This implies that

$$e^{t\Delta_G} = e^{t\|\rho\|^2} e^{t\overline{\text{Duf}}(\Delta_{\mathfrak{g}})}.$$

Then, by Lemma 4.2.33,

$$e^{t\Delta_G} = e^{tS/6} e^{t\overline{\text{Duf}}(\Delta_{\mathfrak{g}})}.$$

This implies that

$$k_t = e^{tS/6} p_t,$$

where  $p_t$  is the heat convolution kernel of  $\overline{\text{Duf}}(\Delta_{\mathfrak{g}})$ . So the asymptotic expansion 4.2.38 follows from Proposition 4.2.31.  $\square$

**4.2.39 COROLLARY.** *Let  $G$  be a compact connected Lie group equipped with a bi-invariant metric. Let  $S$  be the scalar curvature of  $G$ . The heat trace  $Z(t) = \text{tr}(e^{t\Delta_G})$  has the asymptotic expansion*

$$Z(t) \sim \frac{\text{vol}(G)}{(4\pi t)^{\dim G/2}} e^{tS/6} \quad (4.2.40)$$

for  $t \rightarrow 0+$ .

*Proof.* This follows from Theorem 4.2.37 and Equation 2.1.14.  $\square$

*Example.* Let  $G = \text{SU}(2)$ . Take the bi-invariant metric on  $G$  generated by the negative of the Killing form on the Lie algebra of  $G$ . We examined this case in Section 3.2, and calculated that (Equations 3.2.10 and 3.2.23)

$$\|\rho\|^2 = \frac{1}{8}, \quad \text{vol}(G) = 32\sqrt{2}\pi^2.$$

So, by Lemma 4.2.33 and Corollary 4.2.39, the heat trace  $Z(t)$  has the following asymptotic expansion:

$$Z(t) \sim \frac{4\sqrt{2}\pi}{t^{3/2}} e^{t/8}.$$

*Remark.* H. P. McKean and I. M. Singer [74] calculated the asymptotic expansion for the heat trace, using differential geometric methods, up to  $O(t^{3-n/2})$  for any closed Riemannian manifold, where  $n$  is the dimension of the manifold. Our result (Corollary 4.2.39) is consistent with theirs. H. D. Fegan, in [39], obtained a formula for the

exact heat kernel on a simply connected, compact, semisimple Lie group; but S. Zelditch [101] claims that it is erroneous. H. D. Fegan, using his heat kernel formula, also derived the asymptotic expansion for the heat trace; somehow, it is consistent with our result.

# 5

## THE QUANTUM WEIL ALGEBRA

SO far our focus has been on the Laplacian. We now shift our attention to an operator that is in a sense a “square root” of the Laplacian, namely, the Dirac operator. As P. A. M. Dirac [29] noticed, the vector bundle on which the Dirac operator acts cannot, in general, be the trivial line bundle, and we must consider the spinor bundles.

A natural algebraic model for the space of equivariant differential operators acting on the sections of the spinor bundles over homogeneous spaces is the quantum Weil algebra introduced by A. Alekseev and E. Meinrenken [2, 3]. This chapter is a brief account on their theory.

As usual,  $G$  denotes a compact connected Lie group with Lie algebra  $\mathfrak{g}$ , equipped with an invariant inner product on  $\mathfrak{g}$  (that is, the  $\text{ad}(\mathfrak{g})$ -action is antisymmetric).

*Remark.* Following the custom, we shall call a differential operator on the space of sections of a vector bundle  $F$  as a *differential operator on the vector bundle  $F$* .

### 5.1 CLIFFORD ALGEBRAS

5.1.1 A prerequisite for understanding the quantum Weil algebra is the theory of Clifford algebras. Let us go over some relevant facts and the conventions we shall use.

5.1.2 CLIFFORD ALGEBRA OF AN INNER PRODUCT SPACE. Let  $V$  be a vector space over  $\mathbb{R}$  equipped with an inner product  $\langle \cdot, \cdot \rangle$ . Let  $T(V)$  be the tensor algebra of  $V$ . Let  $J(V)$  denote the ideal in  $T$  generated by the elements of the form

$$vw + wv - \langle v, w \rangle. \quad (5.1.3)$$

The *Clifford algebra* generated by  $(V, \langle \cdot, \cdot \rangle)$  is

$$\text{Cl}(V) := T(V)/J(V).$$

There is a universal property that the Clifford algebra inherits from the tensor algebra; namely, if there is a linear map  $\varphi$  from  $V$  to a unital (associative) algebra  $A$  over  $\mathbb{R}$  that satisfies  $\varphi(v)\varphi(v) = \langle v, v \rangle/2$ , then there is a unique algebra homomorphism  $\tilde{\varphi} : \text{Cl}(V) \rightarrow A$  such that  $\varphi = \tilde{\varphi} \circ i_V$  where  $i_V$  is the inclusion map of  $V$  into  $\text{Cl}(V)$ . By the usual argument for universal objects,  $\text{Cl}(V)$  is isomorphic to the Clifford algebra  $\text{Cl}(n)$  generated by the Euclidean space  $\mathbb{R}^n$ .

*Remark.* Note that the vectors  $v$  and  $w$  in  $V$ , as generators of  $\text{Cl}(V)$ , satisfy the relation

$$vw + wv = \langle v, w \rangle. \quad (5.1.4)$$

The traditional convention is to use the relation

$$vw + wv = \epsilon \langle v, w \rangle$$

with  $\epsilon$  equal to 2 or  $-2$ . The specific choice for the value of  $\epsilon$  bears no weight for our purposes. We do point out, though, that because we chose  $\epsilon = 1$ , the top order part (or the principal symbol) of the square of the Dirac operator will be equal to that of the Laplacian times  $1/2$ .

5.1.5 SPINORS. We define the complexification of  $\text{Cl}(V)$  as

$$\mathbb{C}\text{Cl}(V) := \text{Cl}(V) \otimes \mathbb{C}.$$

Since  $\text{Cl}(V) \simeq \text{Cl}(n)$  for some nonnegative integer  $n$ , we have

$$\mathbb{C}\text{Cl}(V) \simeq \text{Cl}(n) \otimes \mathbb{C} =: \mathbb{C}\text{Cl}(n).$$

According to the general theory of Clifford algebras (see [43, p. 13]),

$$\mathbb{C}\text{Cl}(n) \simeq \begin{cases} \text{End}_{\mathbb{C}}(\mathbb{C}^{2^k}), & \text{if } n = 2k; \\ \text{End}_{\mathbb{C}}(\mathbb{C}^{2^k}) \oplus \text{End}_{\mathbb{C}}(\mathbb{C}^{2^k}), & \text{if } n = 2k + 1. \end{cases} \quad (5.1.6)$$

These isomorphisms give rise to an irreducible representation of  $\mathbb{C}\text{Cl}(n)$  on the  $2^k$ -dimensional complex vector space with  $k = \lfloor n/2 \rfloor$ . This representation space is called the space of *spinors* for  $\mathbb{C}\text{Cl}(n)$  (or *n-spinors*, for short); we shall denote it by  $\mathbb{S}_n$ . It is clear from the isomorphism 5.1.6 that, when  $n = 2k$ , there is only one way  $\mathbb{C}\text{Cl}(2k)$  can act on  $\mathbb{S}_{2k}$ . As a consequence, any  $\mathbb{C}\text{Cl}(2k)$ -module  $E$  is of the form

$$E = \mathbb{S}_{2k} \otimes W \quad (5.1.7)$$

where  $W$  is some auxiliary complex vector space on which  $\mathbb{C}\text{Cl}(2k)$  acts trivially; put in another way,

$$W \simeq \text{Hom}_{\mathbb{C}\text{Cl}(2k)}(\mathbb{S}_{2k}, E).$$

When  $n = 2k + 1$ , it turns out that there are two ways  $\text{Cl}(2k + 1)$  can act on  $\mathbb{S}_{2k+1}$ ; see [43, p. 13] for details.

5.1.8 Let  $\{e_i\}_{i=1}^n$ ,  $n = \dim(V)$ , be an orthonormal basis for  $V$ . Then the simple tensors of the form  $e_{i_1} \otimes \cdots \otimes e_{i_k}$ ,  $k \in \mathbb{N}$ , together with 1, form a basis for the tensor algebra  $T(V)$ . It is customary to write the image of  $e_{i_1} \otimes \cdots \otimes e_{i_k}$  under the quotient map  $T(V) \rightarrow T(V)/J(V) = \text{Cl}(V)$  as  $e_{i_1} \cdots e_{i_k}$ . Among these images, the ones that are not zero are the following: For each  $I \subseteq \{1, \dots, n\}$ ,

$$e_I := \begin{cases} 1, & \text{for } I = \emptyset; \\ e_{i_1} \cdots e_{i_k}, & \text{for } I = \{i_1, \dots, i_k\} \text{ with } i_1 < \cdots < i_k. \end{cases} \quad (5.1.9)$$

The collection  $\{e_I \mid I \subseteq \{1, \dots, n\}\}$  is a linear basis for  $\text{Cl}(V)$ .

As a consequence, we have an isomorphism  $\text{Cl}(V) \simeq \mathbb{R}^{2^n}$  as vector spaces. This induces a differentiable structure on  $\text{Cl}(V)$ . According to a general fact regarding the group of units in a finite-dimensional algebra (see, for instance, [28, § 16.9.3]) the group  $\text{Cl}(V)^\times$  of units in  $\text{Cl}(V)$  is an open subset of  $\text{Cl}(V)$  and is a Lie group under the inherited submanifold structure.

5.1.10  $\text{Cl}(V)$  AS A FILTERED ALGEBRA. The natural grading on  $T(V)$  induces a filtration on  $\text{Cl}(V)$ , that is,

$$\text{Cl}_k(V) := \left( \bigoplus_{q=-\infty}^k T^q(V) \right) / J(V), \quad k \in \mathbb{Z},$$

gives a filtration for  $\text{Cl}(V)$ . (In the above definition, we set  $T^q(V) = \{0\}$  if  $q$  is negative.) The basis elements  $e_I$  defined by Equation 5.1.9 have filtration order  $|I|$ . Thus,

$$\text{Span}_{\mathbb{R}} \{e_I : |I| \leq k\} = \text{Cl}_k(V). \quad (5.1.11)$$

Moreover, we have a canonical linear isomorphism

$$\begin{aligned} \sigma_k : \text{Span}_{\mathbb{R}} \{e_I : |I| = k\} &\rightarrow \text{Cl}_k(V) / \text{Cl}_{k-1}(V), \\ e_I &\mapsto e_I + \text{Cl}_{k-1}(V), \end{aligned} \quad (5.1.12)$$

for each  $k \in \mathbb{Z}$ ; collectively, we have a vector space isomorphism

$$\sigma : \text{Cl}(V) \rightarrow \bigoplus_{k \in \mathbb{Z}} \text{Cl}_k(V) / \text{Cl}_{k-1}(V). \quad (5.1.13)$$

This is the symbol map (see page 57) from  $\text{Cl}(V)$  to its associated graded algebra  $\text{gr Cl}(V)$ .

5.1.14  $\text{Cl}(V)$  AS A SUPER-ALGEBRA. The tensor algebra  $T(V)$  becomes a super-algebra<sup>1</sup> by reducing its natural  $\mathbb{Z}$ -grading modulo  $2\mathbb{Z}$ . Since the relation 5.1.4 respects this  $\mathbb{Z}/2\mathbb{Z}$ -grading, the Clifford algebra

<sup>1</sup> ‘Super’ is synonymous to ‘ $\mathbb{Z}/2\mathbb{Z}$ -graded’.

$\text{Cl}(V)$  is also a super-algebra:

$$\text{Cl}(V) = \text{Cl}^0(V) \oplus \text{Cl}^1(V).$$

In terms of the basis elements  $e_I$  (Equation 5.1.9), we have

$$\text{Cl}^0(V) = \text{Span}_{\mathbb{R}} \{ e_I : |I| \equiv 0 \pmod{2} \},$$

$$\text{Cl}^1(V) = \text{Span}_{\mathbb{R}} \{ e_I : |I| \equiv 1 \pmod{2} \}.$$

We define the *super-commutator*  $[\cdot, \cdot]_s$  on  $\text{Cl}(V)$  as, for homogeneous elements  $x$  and  $y$ ,

$$[x, y]_s = xy - (-1)^{\deg(x)\deg(y)}yx.$$

**5.1.15 CHEVALLEY IDENTIFICATION.** Let  $\epsilon \in [0, 1]$ . Define  $\text{Cl}(V, \epsilon)$  as the Clifford algebra generated by  $(V, \epsilon \langle \cdot, \cdot \rangle)$ . If  $\epsilon = 1$ , we get back  $\text{Cl}(V)$ . If  $\epsilon = 0$ , then we have the exterior algebra  $\wedge(V)$ .

As before, let  $\{e_i\}_{i=1}^n$  be an orthonormal basis for  $V$  with respect to the original inner product  $\langle \cdot, \cdot \rangle$ . Define, for each  $I \subseteq \{1, \dots, n\}$ , the elements  $e_I^\epsilon$  in  $\text{Cl}(V, \epsilon)$  as

$$e_I^\epsilon := \begin{cases} 1, & \text{for } I = \emptyset; \\ e_{i_1} \cdots e_{i_k}, & \text{for } I = \{i_1, \dots, i_k\} \text{ with } i_1 < \dots < i_k. \end{cases} \quad (5.1.16)$$

These elements form a linear basis for  $\text{Cl}(V, \epsilon)$ . Note that  $e_I^1$  is what we defined earlier as  $e_I$  (Equation 5.1.9). In any case, we have a vector space isomorphism:

$$\begin{aligned} \text{Cl}(V, \epsilon) &\rightarrow \text{Cl}(V), \\ e_I^\epsilon &\mapsto e_I. \end{aligned}$$

When  $\epsilon = 0$ , this vector space isomorphism is

$$\begin{aligned} q : \quad \wedge(V) &\rightarrow \text{Cl}(V), \\ e_{i_1} \wedge \cdots \wedge e_{i_k} &\mapsto e_{i_1} \cdots e_{i_k}. \end{aligned} \quad (5.1.17)$$

That  $q$  is a vector space isomorphism was first stated and proved by C. Chevalley [25, Thm. II.1.6, p. 41]. Following A. Alekseev and E. Meinrenken [3], we call this map the *Chevalley (quantization) map*. Note that, since

$$e_i e_j = -e_j e_i + \delta_{ij}, \quad (5.1.18)$$

where  $\delta_{ij}$  is the Kronecker delta symbol, we have

$$e_{i_1} \cdots e_{i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) e_{\sigma(1)} \cdots e_{\sigma(k)}.$$

The right-hand side is the image of the alternating tensor

$$\frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)} \quad (5.1.19)$$

under the canonical projection  $\pi_{\text{Cl}} : T(V) \rightarrow T(V)/J(V) = \text{Cl}(V)$ . Now the tensors of the form 5.1.19 span the subspace  $\text{Alt}(V)$  of the

alternating tensors in  $T(V)$ . It is a standard fact in algebra (see [31, Prop. 40, p. 453]) that  $\text{Alt}(V)$  is linearly isomorphic to  $\wedge(V)$  by the antisymmetrization map:

$$\text{Alt}(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}. \quad (5.1.20)$$

Here  $v_i$ 's are arbitrary vectors in  $V$ . So we have the following commutative diagram of linear isomorphisms:

$$\begin{array}{ccc} & & \text{Alt}(V) \subseteq T(V) \\ & \nearrow \text{Alt} & \downarrow \pi_{\text{Cl}} \\ \wedge(V) & \xrightarrow{q} & \text{Cl}(V) \end{array}$$

In this sense, the Chevalley map is the antisymmetrization map.

It is reasonable to guess that  $\wedge(\mathfrak{g})$  is isomorphic to  $\text{gr Cl}(V)$  as algebras. Let us check this. The very definition of the Chevalley map 5.1.12 shows that the  $k$ th homogeneous component  $\wedge^k(V)$  of the exterior algebra is mapped bijectively onto  $\text{Span}_{\mathbb{R}}\{e_I : |I| = k\}$  in the Clifford algebra. So the composition of the Chevalley map with the linear isomorphism 5.1.12 gives us the linear isomorphism

$$\begin{aligned} \wedge^k(V) &\xrightarrow{\sigma_k \circ q} \text{Cl}_k(V)/\text{Cl}_{k-1}(V), \\ e_I &\longmapsto e_I + \text{Cl}_{k-1}(V). \end{aligned} \quad (5.1.21)$$

Collectively, we have the linear isomorphism

$$\wedge(V) \xrightarrow{\sigma \circ q} \text{gr Cl}(V). \quad (5.1.22)$$

This isomorphism is, in fact, multiplicative (so it is an algebra isomorphism). It is sufficient to check for the product of two basis elements  $e_{i_1} \wedge \cdots \wedge e_{i_k}$  and  $e_{j_1} \wedge \cdots \wedge e_{j_\ell}$  in  $\wedge(V)$ , where  $1 \leq i_1 < \cdots < i_k \leq n$  and  $1 \leq j_1 < \cdots < j_\ell \leq n$ . The image of these two basis elements under the map 5.1.22 are  $e_{i_1} \cdots e_{i_k} + \text{Cl}_{k-1}(V)$  and  $e_{j_1} \cdots e_{j_\ell} + \text{Cl}_{\ell-1}(V)$ , respectively, and their product is  $e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_\ell} + \text{Cl}_{k+\ell-1}(V)$ . So we need to check whether

$$q(e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge \cdots \wedge e_{j_\ell}) = e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_\ell} \text{ mod } \text{Cl}_{k+\ell-1}(V). \quad (5.1.23)$$

If  $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_\ell\} = \emptyset$ , then  $e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge \cdots \wedge e_{j_\ell}$  is one of the standard basis element (up to sign) for  $\wedge(V)$ , so  $q$  maps it to  $e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_\ell}$ . If  $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_\ell\} \neq \emptyset$ , then  $e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge \cdots \wedge e_{j_\ell} = 0$ , but  $e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_\ell}$  is in  $\text{Cl}_{|I|+|J|-2}(V)$  (owing to the relation  $e_i e_j = -e_j e_i + \delta_{ij}$ ), so Equation 5.1.23 holds. In fact, our calculations show that we have a stronger relation:

$$q(e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge \cdots \wedge e_{j_\ell}) = e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_\ell} \text{ mod } \text{Cl}_{k+\ell-2}(V). \quad (5.1.24)$$

To sum up, we may identify  $\wedge(V)$  with  $\text{gr Cl}(V)$ . Then the Chevalley map is the inverse of the symbol map. Put in another way, we

have the following commutative diagram of vector space isomorphisms:

$$\begin{array}{ccc} & \wedge(V) & \\ q \swarrow & \downarrow \sigma \circ q & \\ \text{Cl}(V) & & \text{gr Cl}(V) \\ \searrow \sigma & & \end{array}$$

**5.1.25 CLIFFORD MULTIPLICATION ON  $\wedge(V)$ .** Since  $\text{Cl}(V)$  and  $\wedge(V)$  are linearly isomorphic, we can transfer the product structure of  $\text{Cl}(V)$  to  $\wedge(V)$ ; we define the *Clifford multiplication* of  $x$  and  $y$  in  $\wedge(V)$  as

$$x \cdot y := q^{-1}(q(x)q(y)).$$

The very fact that that  $\wedge(V)$  is the associated graded algebra of  $\text{Cl}(V)$  implies that the Clifford multiplication is equal to the exterior multiplication modulo terms of lower degree. More precisely, we have, by Equation 5.1.24,

$$x \cdot y = x \wedge y \mod \bigoplus_{q=0}^{k+\ell-2} \wedge^q(V) \quad (5.1.26)$$

for homogeneous elements  $x$  and  $y$  of degree  $k$  and  $\ell$ , respectively.

**5.1.27 DEFINITION.** We define the *Clifford commutator*  $[\cdot, \cdot]_s$  on  $\wedge(V)$  to be the super-commutator with respect to the Clifford multiplication. That means

$$[x, y]_s = x \cdot y - (-1)^{k\ell} y \cdot x$$

for homogeneous elements  $x$  and  $y$  of degree  $k$  and  $\ell$ , respectively.

*Remark.* By Equation 5.1.26, we have

$$[x, y]_s = (x \wedge y - (-1)^{k\ell} y \wedge x) \mod \bigoplus_{q=0}^{k+\ell-2} \wedge^q(V).$$

The exterior algebra is super-commutative in the sense that  $x \wedge y - (-1)^{k\ell} y \wedge x = 0$ ; so

$$[x, y]_s = 0 \mod \bigoplus_{q=0}^{k+\ell-2} \wedge^q(V). \quad (5.1.28)$$

**5.1.29 LIE ALGEBRA ISOMORPHISM  $\wedge^2(V) \simeq \mathfrak{so}(V)$ .** Let  $x$  be a homogeneous element of  $\wedge(V)$  of degree 2. If  $v$  is a vector in  $V = \wedge^1(V)$  then, by Equation 5.1.28,  $[x, v]_s = 0$  modulo  $\wedge^1(V)$ ; so the Clifford commutator  $[x, v]_s$  is again a vector in  $V$ . Hence, each  $x$  in  $\wedge^2(V)$  defines a linear endomorphism on  $V$  by

$$\begin{aligned} \text{ad}_s(x) : V &\rightarrow V, \\ v &\mapsto [x, v]_s. \end{aligned}$$

As a simple example, consider  $x := e_i \wedge e_j$  with  $1 \leq i < j \leq n$ .



Then

$$\begin{aligned} [x, v]_s &= q^{-1}(q(e_i e_j)q(v)) - q^{-1}(q(v)q(e_i e_j)) \\ &= q^{-1}(q(e_i e_j)q(v) - q(v)q(e_i e_j)) \\ &= q^{-1}(e_i e_j v - v e_i e_j). \end{aligned} \quad (5.1.30)$$

From the relation  $e_j v = -v e_j + \langle e_j, v \rangle$ , we get

$$e_i e_j v = -e_i v e_j + \langle e_j, v \rangle e_i = v e_i e_j - \langle e_i, v \rangle e_j + \langle e_j, v \rangle e_i.$$

Inserting this into Equation 5.1.30,

$$\begin{aligned} [e_i \wedge e_j, v]_s &= q^{-1}(\langle e_j, v \rangle e_i - \langle e_i, v \rangle e_j) \\ &= \langle e_j, v \rangle e_i - \langle e_i, v \rangle e_j. \end{aligned} \quad (5.1.31)$$

Hence, the Clifford commutation with  $e_i \wedge e_j$  is a antisymmetric operator on  $V$ . By the bi-linearity of the commutator, we concluded that this is true for Clifford commutation with any element in  $\wedge^2(V)$ .

Therefore, we have an injective linear map

$$\text{ad}_s : \wedge^2(V) \rightarrow \mathfrak{so}(V). \quad (5.1.32)$$

We may asks whether this is surjective. The answer is affirmative; in fact, as B. Kostant shows in [64, Thm. 8, p. 286], this is a Lie algebra isomorphism as we set the Clifford commutator as the Lie bracket for  $\wedge^2(V)$ . A formula for the inverse of the map 5.1.32 is as follows: For  $T$  in  $\mathfrak{so}(V)$ ,

$$\text{ad}_s^{-1}(T) = \frac{1}{2} \sum_{i=1}^n T(v_i) \wedge v^i, \quad (5.1.33)$$

where  $\{v_i\}_{i=1}^n$  and  $\{v^i\}_{i=1}^n$  are two basis for  $V$  that are dual with respect to the inner product so that  $\langle v_i, v^j \rangle = \delta_{ij}$ . This formula does not depend on the choice for the basis.

**5.1.34 DEFINITION.** A *super-derivation* on a super-algebra  $A$  is a linear map  $\delta : A \rightarrow A$  such that

$$\delta(xy) = \delta(x)y + (-1)^{\deg(x)n} x\delta(y)$$

for homogeneous elements  $x$  and  $y$  in  $A$ , where  $n$  is any integer. The derivation is said to be *even* or *odd*, according to the parity of  $n$ .

The following lemma is from [64, Lem. 5, p. 284]:

**5.1.35 LEMMA.** Let  $\iota_v$  denote the interior product on  $\wedge(V)$  with respect to  $v$ , that is, the odd derivation on  $\wedge(V)$  generated by

$$\iota_v u = \langle u, v \rangle \quad (5.1.36)$$

for  $u \in \wedge^1(V)$ . Then, for  $x$  in  $\wedge^2(V)$ ,

$$[x, v]_s = -\iota_v x. \quad (5.1.37)$$

With slight abuse of notation, let  $\iota_v$  also denote the odd derivation on

$\text{Cl}(V)$  generated by Equation 5.1.36. Then,

$$[qx, v]_s = -\iota_v(qx). \quad (5.1.38)$$

*Proof.* Let  $\{e_i\}_{i=1}^{\dim V}$  be an orthonormal basis for  $V$ . Owing to linearity, it is sufficient to check Equation 5.1.37 for the case  $x = e_i \wedge e_j$ ; but that is done by Equation 5.1.31.

According to the definition of the Clifford commutator, Equation 5.1.37 means

$$q^{-1}[qx, qv]_s = -\iota_v x$$

where  $[\ , \ ]_s$  now represents the super-commutator on  $\text{Cl}(V)$ . So we have

$$[qx, qv]_s = -q(\iota_v x).$$

Since  $qu = u$  for any  $u$  in  $V$ , the above gives us

$$[qx, v]_s = -\iota_v x.$$

Hence, to prove Equation 5.1.38, it is left to show that

$$\iota_v x = \iota_v(qx). \quad (5.1.39)$$

Again, it is sufficient to check for the case  $x = e_i \wedge e_j$ . For  $\iota_v(qx)$ , we have

$$\iota_v(q(e_i \wedge e_j)) = \iota_v(e_i e_j).$$

Then, by the odd derivation property of  $\iota_v$ ,

$$\iota_v(q(e_i \wedge e_j)) = \iota_v(e_i) e_j - e_i \iota_v(e_j) = \langle e_i, v \rangle e_j - \langle e_j, v \rangle e_i = \iota_v(e_i \wedge e_j). \quad \square$$

**5.1.40 SPIN GROUP.** We define the *spin group*  $\text{Spin}(V)$  and its Lie algebra  $\mathfrak{spin}(V)$  as follows. (This is due to B. Kostant; see § 2, especially Thm. 8, in [64].) Consider the inverse of the map 5.1.32,

$$\text{ad}_s^{-1} : \mathfrak{so}(V) \xrightarrow{\sim} \wedge^2(V). \quad (5.1.41)$$

As we have noted in Section 5.1.29, this is a Lie algebra isomorphism where the Lie bracket on  $\wedge^2(V)$  is given by the Clifford commutator. The Clifford commutator on  $\wedge(V)$  was defined so that the Chevalley map  $q$  intertwines it with the super-commutator on  $\text{Cl}(V)$ ; thus, composing  $\text{ad}_s^{-1}$  with  $q$  yields a Lie-algebra isomorphism of  $\mathfrak{so}(V)$  onto its image in  $\text{Cl}(V)$ ; this image is the Lie algebra  $\mathfrak{spin}(V)$ .

$$\mathfrak{so}(V) \xrightarrow[\text{q} \circ \text{ad}_s^{-1}]{\sim} \mathfrak{spin}(V) \subseteq \text{Cl}(V). \quad (5.1.42)$$

Let  $\{e_i\}_{i=1}^{\dim V}$  be an orthonormal basis for  $V$ . Since the set

$$\{e_i \wedge e_j \mid 1 \leq i < j \leq \dim V\}$$

spans  $\wedge^2(V)$ , its image under the quantization map

$$\{e_i e_j \mid 1 \leq i < j \leq \dim V\}$$

spans  $\mathfrak{spin}(V)$ .

Now let  $\exp_{\text{Cl}}$  denote the exponentiation in  $\text{Cl}(V)$ ; that is, for  $x$  in  $\text{Cl}(V)$ ,

$$\exp_{\text{Cl}}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then,

$$\text{Spin}(V) := \exp_{\text{Cl}}(\mathfrak{spin}(V)).$$

The Clifford exponential map  $\exp_{\text{Cl}}$  on  $\mathfrak{spin}(V)$  agrees with the Lie-theoretic exponential map  $\exp : \mathfrak{spin}(V) \rightarrow \text{Spin}(V)$ , owing to the uniqueness of 1-parameter subgroups.

As an example, let us calculate  $\exp_{\text{Cl}}(te_i e_j)$  where  $t$  is a parameter in  $\mathbb{R}$ . Note that

$$(e_i e_j)^2 = e_i e_j e_i e_j = -e_i e_i e_j e_j = -\frac{1}{2^2}.$$

Hence,

$$\begin{aligned} \exp_{\text{Cl}}(te_i e_j) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{t}{2}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{t}{2}\right)^{2k+1} 2e_i e_j \\ &= \cos(t/2) + 2 \sin(t/2) e_i e_j. \end{aligned} \quad (5.1.43)$$

As our discussion in Section 5.1.29 shows, for each  $x$  in  $\mathfrak{spin}(V)$ , the inner derivation it defines on  $\text{Cl}(V)$  (with respect to the super-commutator) restricts to a skew-symmetric operator on  $V$ . Let us denote this representation by  $\text{ad}_c : \mathfrak{spin}(V) \rightarrow \mathfrak{so}(V)$ . Now define the Lie group representation  $\text{Ad}_c : \text{Spin}(V) \rightarrow \text{Aut}(V)$  by

$$\text{Ad}_c(g)(v) = gvg^{-1},$$

where  $gvg^{-1}$  is the product of  $g$ ,  $v$ , and  $g^{-1}$  as elements of  $\text{Cl}(V)$ . Then,  $\text{ad}_c$  is the differential of  $\text{Ad}_c$ , and we have the following commutative diagram:

$$\begin{array}{ccc} \text{Spin}(V) & \xrightarrow{\text{Ad}_c} & \text{SO}(V) \\ \exp_{\text{Cl}} \uparrow & & \uparrow \exp \\ \mathfrak{spin}(V) & \xrightarrow{\text{ad}_c} & \mathfrak{so}(V) \end{array}$$

Because  $\text{ad}_c$  is a Lie algebra isomorphism,  $\text{Ad}_c$  is a covering map (see [92, Thm. 3.25, p. 100]). This covering is evidently not an isomorphism since  $-1$  in  $\text{Spin}(V)$  acts trivially on  $v$ . In fact, it is a double covering; see [43, Prop. (c), p. 16]. Since  $\text{SO}(V)$  is compact (see [34, § 1.2.A]) and  $\text{Ad}_c$  is a covering map with finite number of sheets,  $\text{Spin}(V)$  is a compact Lie group.

**5.1.44** Because  $\text{Spin}(V)$  is a subgroup of  $\text{Cl}(V)^\times$ , which is contained in  $\text{Cl}(V)$ , the spinor space  $\mathbb{S}$  for  $\text{Cl}(V)$  is a complex representation space for  $\text{Spin}(V)$ . This representation is known as the *spinor representation* of  $\text{Spin}(V)$ . It is a classical result that  $\mathbb{S}$ , as a  $\text{Spin}(V)$ -

vector space, may be reducible or not, depending on the parity of the dimension of  $V$ ; if the dimension is odd, then  $\mathbb{S}$  is irreducible; if the dimension is even, then  $\mathbb{S}$  is completely reducible to

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-, \quad (5.1.45)$$

where the irreducible subspaces  $\mathbb{S}^\pm$  are the eigenspaces with eigenvalues  $\pm(2i)^{-k}$ , respectively, for the action of the element  $e_1 e_2 \cdots e_n$  where  $n = \dim(V)$  (see [43, p. 22] for a proof).

Since  $\text{Spin}(V)$  is compact, the spinor space  $\mathbb{S}$  admits a  $\text{Spin}(V)$ -invariant inner product. In fact, we can expect more; we may assume that the  $\text{Spin}(V)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  is skew-invariant with respect to the action of  $V$  in  $\mathbb{C}l(V)$ , that is, for  $v \in V$  and  $(\xi_1, \xi_2) \in \mathbb{S} \times \mathbb{S}$ ,

$$\langle v \cdot \xi_1, \xi_2 \rangle = -\langle \xi_1, v \cdot \xi_2 \rangle.$$

Moreover, when  $V$  is even-dimensional, the decomposition 5.1.45 is an orthogonal decomposition with respect to this inner product. See [43, p. 24] for details.

**5.1.46 DEFINITION.** Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra with an inner product  $\langle \cdot, \cdot \rangle$  that is skew-invariant under the  $\text{ad}(\mathfrak{g})$ -action. Then the adjoint representation yields a Lie algebra homomorphism

$$\mathfrak{g} \xrightarrow{\text{ad}} \mathfrak{so}(\mathfrak{g}).$$

We shall denote the composition of the above map with the Lie algebra isomorphisms 5.1.41 and 5.1.42, respectively, by

$$\begin{aligned} \lambda : \mathfrak{g} &\rightarrow \wedge^2(\mathfrak{g}), \\ \gamma : \mathfrak{g} &\rightarrow \mathfrak{spin}(\mathfrak{g}). \end{aligned}$$

In other words, for  $X$  in  $\mathfrak{g}$ ,

$$\lambda(X) = \text{ad}_s^{-1}(\text{ad}_X), \quad (5.1.47)$$

$$\gamma(X) = q \circ \lambda(X). \quad (5.1.48)$$

*Remark.* The maps  $\lambda$  and  $\gamma$  are clearly Lie algebra homomorphisms. That means

$$\lambda([X, Y]) = [\lambda(X), \lambda(Y)]_s, \quad (5.1.49)$$

$$\gamma([X, Y]) = [\gamma(X), \gamma(Y)]_s. \quad (5.1.50)$$

**5.1.51 LEMMA.** Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra over  $\mathbb{R}$  with an inner product  $\langle \cdot, \cdot \rangle$  that is skew-invariant under the  $\text{ad}(\mathfrak{g})$ -action. Let  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  and  $\{X^i\}_{i=1}^{\dim \mathfrak{g}}$  be two basis for  $\mathfrak{g}$  that are dual with respect to the inner product so that  $\langle X_i, X^j \rangle = \delta_{ij}$ . In the following,  $X$  and  $Y$  are vectors in  $\mathfrak{g}$ ,  $[X, Y]_{\mathfrak{g}}$  denotes their Lie bracket in  $\mathfrak{g}$ .

(a) The Lie algebra homomorphism  $\lambda$  satisfies

$$\text{ad}_X(Y) = [\lambda(X), Y]_s, \quad (5.1.52)$$

where  $[\cdot, \cdot]_s$  is the Clifford commutator in  $\wedge(\mathfrak{g})$ . And

$$\lambda(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle X, [X_i, X_j]_{\mathfrak{g}} \rangle X^i \wedge X^j. \quad (5.1.53)$$

(b) The Lie algebra homomorphism  $\gamma$  satisfies

$$\text{ad}_X(Y) = [\gamma(X), Y]_s, \quad (5.1.54)$$

where  $[\cdot, \cdot]_s$  is the super-commutator in  $\text{Cl}(\mathfrak{g})$ . And

$$\gamma(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle X, [X_i, X_j]_{\mathfrak{g}} \rangle X^i X^j. \quad (5.1.55)$$

*Proof.* By definition,  $\lambda(X) = \text{ad}_s^{-1}(\text{ad}_X)$ . Now  $\text{ad}_s^{-1}$  was defined so that, for any  $T \in \mathfrak{so}(\mathfrak{g})$ , we would have  $T(Y) = [\text{ad}_s^{-1}(T), Y]_s$ . Setting  $T = \text{ad}_X$ , we get

$$\text{ad}_X(Y) = [\text{ad}_s^{-1}(\text{ad}_X), Y]_s = [\lambda(X), Y]_s.$$

This proves Equation 5.1.52. Let us deduce Equation 5.1.54 from this. Applying the Chevalley map  $q$  to the above equation, we get

$$q(\text{ad}_X(Y)) = q[\lambda(X), Y]_s.$$

The left-hand side is just  $\text{ad}_X(Y)$ . For the right-hand side, recall that the Chevalley map  $q$  intertwines the Clifford commutator on  $\wedge(\mathfrak{g})$  with the super-commutator on  $\text{Cl}(\mathfrak{g})$ ; so

$$q[\lambda(X), Y]_s = [q\lambda(X), qY]_s = [\gamma(X), Y]_s.$$

Here the first commutator is the Clifford commutator in  $\wedge(\mathfrak{g})$ , and the other two commutators are the super-commutator in  $\text{Cl}(\mathfrak{g})$ . This proves Equation 5.1.54.

Next, let us verify Equation 5.1.53. Again, begin with the definition  $\lambda(X) = \text{ad}_s^{-1}(\text{ad}_X)$ . Then, by Equation 5.1.33,

$$\lambda(X) = \frac{1}{2} \sum_{j=1}^{\dim \mathfrak{g}} \text{ad}(X)(X_j) \wedge X^j = \frac{1}{2} \sum_{j=1}^{\dim \mathfrak{g}} [X, X_j]_{\mathfrak{g}} \wedge X^j. \quad (5.1.56)$$

Now, for any vector  $Y$  in  $\mathfrak{g}$ , we have  $Y = \sum_{i=1}^{\dim \mathfrak{g}} \langle Y, X_i \rangle X^i$ . So

$$[X, X_j]_{\mathfrak{g}} = \sum_{i=1}^{\dim \mathfrak{g}} \langle [X, X_j]_{\mathfrak{g}}, X_i \rangle X^i.$$

Inserting this into Equation 5.1.56, we get

$$\lambda(X) = \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle [X, X_i]_{\mathfrak{g}}, X_j \rangle X^i \wedge X^j.$$

By the invariance of the inner product, the above can be rewritten

as

$$\lambda(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle X, [X_i, X_j]_{\mathfrak{g}} \rangle X^i \wedge X^j.$$

This proves Equation 5.1.53. Applying the Chevalley map to both sides of this equation gives us Equation 5.1.55.  $\square$

## 5.2 THE QUANTUM WEIL ALGEBRA

5.2.1 For the rest of this chapter,  $\mathfrak{g}$  is a finite-dimensional Lie algebra over  $\mathbb{R}$  equipped with an invariant inner product  $\langle \cdot, \cdot \rangle$ . (For example,  $\mathfrak{g}$  is the Lie algebra of a compact Lie group endowed with a bi-invariant metric.) Let  $\bar{\mathfrak{g}}$  be a copy of  $\mathfrak{g}$ , and for each  $X$  in  $\mathfrak{g}$ , denote the corresponding vector in  $\bar{\mathfrak{g}}$  by  $\bar{X}$ . The starting point of A. Alekseev and E. Meinrenken's theory is to consider the super-space (that is, the  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space)

$$\mathfrak{g} \oplus \bar{\mathfrak{g}},$$

where  $\mathfrak{g}$  is the subspace of odd degree and  $\bar{\mathfrak{g}}$  is the subspace of even degree. The *quantum Weil algebra*  $\mathcal{W}(\mathfrak{g})$  is the unital super-algebra over  $\mathbb{R}$  generated by the super-space  $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ , subject to the following super-commutator relations:

$$[X, Y]_s = \langle X, Y \rangle, \quad (5.2.2)$$

$$[\bar{X}, Y]_s = [X, Y]_{\mathfrak{g}}, \quad (5.2.3)$$

$$[\bar{X}, \bar{Y}]_s = \overline{[X, Y]_{\mathfrak{g}}}. \quad (5.2.4)$$

Here  $[\cdot, \cdot]_{\mathfrak{g}}$  denotes the Lie bracket in  $\mathfrak{g}$ . Note that  $\mathfrak{g}$  alone generates the Clifford algebra  $\text{Cl}(\mathfrak{g})$  and that  $\bar{\mathfrak{g}}$  alone generates the universal enveloping algebra  $\mathcal{U}(\bar{\mathfrak{g}})$ . This takes into account Equations 5.2.2 and 5.2.3. Equation 5.2.4 generates a  $\mathcal{U}(\bar{\mathfrak{g}})$ -action on  $\text{Cl}(\mathfrak{g})$ . So, as a vector space,  $\mathcal{W}(\mathfrak{g})$  is isomorphic to  $\text{Cl}(\mathfrak{g}) \otimes \mathcal{U}(\bar{\mathfrak{g}})$ . As an algebra,  $\mathcal{W}(\mathfrak{g})$  is a semi-direct product:

$$\mathcal{W}(\mathfrak{g}) = \text{Cl}(\mathfrak{g}) \rtimes \mathcal{U}(\bar{\mathfrak{g}}).$$

Each vector  $X$  in  $\mathfrak{g}$  generates two inner derivations on  $\mathcal{W}(\mathfrak{g})$ :

$$\iota_X(\cdot) = [X, \cdot]_s, \quad L_X(\cdot) = [\bar{X}, \cdot]_s. \quad (5.2.5)$$

These are called the *contraction* and the *Lie derivative* by  $X$ , respectively. The contraction is an odd derivation whereas the Lie derivative is an even one. In terms of the generators, we have

$$\iota_X Y = \langle X, Y \rangle, \quad \iota_X \bar{Y} = [X, Y]_{\mathfrak{g}}, \quad (5.2.6)$$

$$L_X Y = [X, Y]_{\mathfrak{g}}, \quad L_X \bar{Y} = \overline{[X, Y]_{\mathfrak{g}}}. \quad (5.2.7)$$

Augment this collection of derivations with a differential (that is a graded derivation of degree 1) defined by

$$dX = \bar{X}, \quad d\bar{X} = 0. \quad (5.2.8)$$

The contractions, the Lie derivatives, and the differential satisfy the following super-commutator relations [2, Thm. 3.6, p. 145; 3, § 3]:

$$\begin{aligned} [d, d]_s &= 0, & [\iota_X, d]_s &= L_X, & [L_X, d]_s &= 0, \\ [L_X, L_Y]_s &= L_{[X, Y]}, & [L_X, \iota_Y]_s &= \iota_{[X, Y]}, & [\iota_X, \iota_Y]_s &= 0. \end{aligned}$$

A. Alekseev and E. Meinrenken [3] refers to this as a *g-differential algebra* structure (*g-da* for short).

A remarkable fact found by A. Alekseev and E. Meinrenken [3, Prop. 5.1, p. 317] is that the differential is also inner; there is an element  $\mathcal{D}$  in  $\mathcal{W}(\mathfrak{g})$  such that

$$d(\cdot) = [\mathcal{D}, \cdot]_s. \quad (5.2.9)$$

In terms of an orthonormal basis  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  for  $\mathfrak{g}$ ,

$$\mathcal{D} = \sum_{i=1}^{\dim \mathfrak{g}} X_i \left( \bar{X}_i - \frac{2}{3} \gamma(X_i) \right), \quad (5.2.10)$$

where  $\gamma(X_k)$  is the image of  $X_k$  under the map  $\gamma$  defined in Definition 5.1.46. Because the differential is independent of the choice for the orthonormal basis, Equation 5.2.9 shows that  $\mathcal{D}$  is also independent of the choice for the orthonormal basis. The element  $\mathcal{D}$  is called the *cubic “Dirac operator”* (cubic with respect to the filtration for  $\mathcal{W}(\mathfrak{g})$  which we shall describe in Section 5.3.1) or the *Kostant-Dirac operator* in  $\mathcal{W}(\mathfrak{g})$ .

Another significance of the element  $\mathcal{D}$  lies in the formula for its square; see Equation 5.2.23.

**5.2.11 CHANGE OF (EVEN) GENERATORS.** Equation 5.1.54 states that

$$[X, Y]_{\mathfrak{g}} = [\gamma(X), Y]_s.$$

According to Equation 5.2.3, the left-hand side is equal to  $[\bar{X}, Y]_s$ . So we have

$$[\bar{X} - \gamma(X), Y]_s = 0. \quad (5.2.12)$$

The element

$$\hat{X} := \bar{X} - \gamma(X)$$

is an even element in  $\mathcal{W}(\mathfrak{g})$ ; it can replace  $\bar{X}$  as an even generator. Denote the new set of even generators by

$$\hat{\mathfrak{g}} := \{\hat{X} \mid X \in \mathfrak{g}\} \subseteq \mathcal{W}(\mathfrak{g}).$$

The super-commutator relations satisfied by the generators in  $\mathfrak{g}$  and  $\hat{\mathfrak{g}}$  are

$$[X, Y]_s = \langle X, Y \rangle, \quad (5.2.13)$$

$$[\hat{X}, Y]_s = 0, \quad (5.2.14)$$

$$[\hat{X}, \hat{Y}]_s = \widehat{[X, Y]}_{\mathfrak{g}}. \quad (5.2.15)$$

Equation 5.2.13 is from Equation 5.2.2, and Equation 5.2.14 is just Equation 5.2.12. So we only need to verify Equation 5.2.15. Since  $\hat{X} = \bar{X} - \gamma(X)$  and  $\hat{Y} = \bar{Y} - \gamma(Y)$ , we have

$$[\hat{X}, \hat{Y}]_s = [\bar{X}, \bar{Y}]_s - [\gamma(X), \bar{Y}]_s - [\bar{X}, \gamma(Y)]_s + [\gamma(X), \gamma(Y)]_s.$$

Then, by Equations 5.1.50 and 5.2.4,

$$[\hat{X}, \hat{Y}]_s = \overline{[X, Y]}_{\mathfrak{g}} - [\gamma(X), \bar{Y}]_s - [\bar{X}, \gamma(Y)]_s + \gamma([X, Y]_{\mathfrak{g}}).$$

It is left to check that

$$-[\gamma(X), \bar{Y}]_s - [\bar{X}, \gamma(Y)]_s = -2\gamma([X, Y]_{\mathfrak{g}}). \quad (5.2.16)$$

By Equation 5.1.55,

$$[\bar{X}, \gamma(Y)]_s = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle Y, [X_i, X_j]_{\mathfrak{g}} \rangle ([\bar{X}, X^i]_s X^j + X^i [\bar{X}, X^j]_s).$$

Applying Equation 5.2.3 to the right-hand side,

$$[\bar{X}, \gamma(Y)]_s = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle Y, [X_i, X_j]_{\mathfrak{g}} \rangle ([X, X^i]_{\mathfrak{g}} X^j + X^i [X, X^j]_{\mathfrak{g}}).$$

Since  $Z = \sum_{k=1}^{\dim \mathfrak{g}} \langle Z, X_k \rangle X^k$  holds for any vector  $Z$  in  $\mathfrak{g}$ , we have

$$\begin{aligned} [\bar{X}, \gamma(Y)]_s &= -\frac{1}{2} \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle Y, [X_i, X_j]_{\mathfrak{g}} \rangle \langle [X, X^i]_{\mathfrak{g}}, X_k \rangle X^k X^j \\ &\quad - \frac{1}{2} \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle Y, [X_i, X_j]_{\mathfrak{g}} \rangle \langle [X, X^j]_{\mathfrak{g}}, X_k \rangle X^i X^k. \end{aligned}$$

By the invariance of the inner product,

$$\begin{aligned} [\bar{X}, \gamma(Y)]_s &= -\frac{1}{2} \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle [Y, X_j]_{\mathfrak{g}}, X_i \rangle \langle X^i, [X, X_k]_{\mathfrak{g}} \rangle X^k X^j \\ &\quad + \frac{1}{2} \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle [Y, X_i]_{\mathfrak{g}}, X_j \rangle \langle X^j, [X, X_k]_{\mathfrak{g}} \rangle X^i X^k. \end{aligned}$$



Using the fact that  $\langle Z, W \rangle = \sum_{i=1}^{\dim \mathfrak{g}} \langle Z, X_i \rangle \langle X^i, W \rangle$  for any vectors  $Z$  and  $W$  in  $\mathfrak{g}$ ,

$$\begin{aligned} [\bar{X}, \gamma(Y)]_s &= -\frac{1}{2} \sum_{j,k=1}^{\dim \mathfrak{g}} \langle [Y, X_j]_{\mathfrak{g}}, [X, X_k]_{\mathfrak{g}} \rangle X^k X^j \\ &\quad + \frac{1}{2} \sum_{i,k=1}^{\dim \mathfrak{g}} \langle [Y, X_i]_{\mathfrak{g}}, [X, X_k]_{\mathfrak{g}} \rangle X^i X^k. \end{aligned}$$

By symmetry, we may exchange  $X$  with  $Y$ , which gives

$$\begin{aligned} [\bar{Y}, \gamma(X)]_s &= -\frac{1}{2} \sum_{j,k=1}^{\dim \mathfrak{g}} \langle [X, X_j]_{\mathfrak{g}}, [Y, X_k]_{\mathfrak{g}} \rangle X^k X^j \\ &\quad + \frac{1}{2} \sum_{i,k=1}^{\dim \mathfrak{g}} \langle [X, X_i]_{\mathfrak{g}}, [Y, X_k]_{\mathfrak{g}} \rangle X^i X^k. \end{aligned}$$

Therefore,

$$\begin{aligned} -[\gamma(X), \bar{Y}]_s - [\bar{X}, \gamma(Y)]_s \\ = \sum_{j,k=1}^{\dim \mathfrak{g}} (\langle [X, X_j]_{\mathfrak{g}}, [Y, X_k]_{\mathfrak{g}} \rangle - \langle [X, X_k]_{\mathfrak{g}}, [Y, X_j]_{\mathfrak{g}} \rangle) X^j X^k. \end{aligned} \quad (5.2.17)$$

Meanwhile,

$$-2\gamma([X, Y]_{\mathfrak{g}}) = \sum_{i,j}^{\dim \mathfrak{g}} \langle [X, Y]_{\mathfrak{g}}, [X_i, X_j]_{\mathfrak{g}} \rangle X^i X^j = \sum_{i,j}^{\dim \mathfrak{g}} \langle [[X, Y]_{\mathfrak{g}}, X_i]_{\mathfrak{g}}, X_j \rangle X^i X^j.$$

By the Jacobi identity,

$$\begin{aligned} -2\gamma([X, Y]_{\mathfrak{g}}) &= \sum_{i,j}^{\dim \mathfrak{g}} (\langle [[X, X_i]_{\mathfrak{g}}, Y]_{\mathfrak{g}}, X_j \rangle + \langle [X, [Y, X_i]_{\mathfrak{g}}]_{\mathfrak{g}}, X_j \rangle) X^i X^j \\ &= \sum_{i,j}^{\dim \mathfrak{g}} (\langle [X, X_i]_{\mathfrak{g}}, [Y, X_j]_{\mathfrak{g}} \rangle - \langle [Y, X_i]_{\mathfrak{g}}, [X, X_j]_{\mathfrak{g}} \rangle) X^i X^j. \end{aligned} \quad (5.2.18)$$

Comparing Equations 5.2.17 and 5.2.18 shows that Equation 5.2.16 holds, which proves Equation 5.2.15.

5.2.19 Because the Lie algebra  $(\hat{\mathfrak{g}}, [\cdot, \cdot]_s)$  is isomorphic to  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ , the even generators in  $\hat{\mathfrak{g}}$  generate the universal enveloping algebra  $\mathcal{U}(\hat{\mathfrak{g}})$  inside  $\mathcal{W}(\mathfrak{g})$ . Since the elements in this subalgebra commutes with the Clifford algebra  $\text{Cl}(\mathfrak{g})$  generated by the odd generators  $X \in \mathfrak{g}$ , we have

$$\mathcal{W}(\mathfrak{g}) = \text{Cl}(\mathfrak{g}) \otimes \mathcal{U}(\hat{\mathfrak{g}}). \quad (5.2.20)$$

5.2.21 The cubic Dirac operator, in terms of the new generators  $\widehat{X}$  and  $X$ , takes the following form:

$$\mathcal{D} = \sum_{i=1}^{\dim \mathfrak{g}} X_i \left( \widehat{X}_i + \frac{1}{3} \gamma(X_i) \right). \quad (5.2.22)$$

Verifying this is just a matter of following the definition; by the definition of  $\widehat{X}_i$ , we have  $\overline{X}_i = \widehat{X}_i - \gamma(X_i)$ ; inserting this into Equation 5.2.10 immediately gives us Equation 5.2.22.

A. Alekseev and E. Meinrenken [2, Prop. 3.4, p. 144] showed that

$$\mathcal{D}^2 = \frac{1}{2} \widehat{\Omega}_{\mathfrak{g}} + \frac{1}{48} \operatorname{tr}_{\mathfrak{g}} \widehat{\Omega}_{\mathfrak{g}}. \quad (5.2.23)$$

Here  $\widehat{\Omega}_{\mathfrak{g}}$  denotes the Casimir element in  $\mathcal{U}(\widehat{\mathfrak{g}})$ , and  $\operatorname{tr}_{\mathfrak{g}} \widehat{\Omega}_{\mathfrak{g}}$  denotes the trace for the adjoint action of  $\widehat{\Omega}_{\mathfrak{g}}$  on  $\mathfrak{g}$ . In terms of an orthonormal basis  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  for  $\mathfrak{g}$ , we have  $\widehat{\Omega}_{\mathfrak{g}} = \sum_{i=1}^{\dim \mathfrak{g}} \widehat{X}_i \widehat{X}_i$  (Equation 2.2.16). We used the caret symbol so that  $\widehat{\Omega}_{\mathfrak{g}}$  would not be confused with  $\sum_{i=1}^{\dim \mathfrak{g}} \overline{X}_i \overline{X}_i$  in  $\{1\} \otimes \mathcal{U}(\widehat{\mathfrak{g}}) \subseteq \mathcal{W}(\mathfrak{g})$ .

*Remark.* A more general form of  $\mathcal{D}$  was independently introduced by B. Kostant in [65], where he also calculates  $\mathcal{D}^2$ . (We shall have a chance to see them later on; see Equation 5.4.18 and Proposition 6.3.14.) The reason  $\mathcal{D}$  is called the “Dirac operator” has to do with the fact that the elements of  $\mathcal{W}(\mathfrak{g}) = \operatorname{Cl}(\mathfrak{g}) \otimes \mathcal{U}(\widehat{\mathfrak{g}})$  can be interpreted as differential operators acting on the sections of the trivial bundle  $G \times E \rightarrow G$  where  $E$  is a finite-dimensional  $\operatorname{Cl}(\mathfrak{g})$ -module; this is obvious for us since we already gave an identification of  $\mathcal{U}(\widehat{\mathfrak{g}})$  as the algebra  $D(G)$  of left-invariant differential operators on  $G$ , via the algebra isomorphism 2.2.11. To recapitulate, for  $X$  in  $\mathfrak{g}$ , the element  $\widehat{X}$  in  $\mathcal{U}(\widehat{\mathfrak{g}})$  is identified as the directional derivative  $\partial_{\widetilde{X}}$  on  $G$  with respect to the left-invariant vector field  $\widetilde{X}$  on  $G$  generated by  $X$ . Now, one might ask, why not identify the generator  $\overline{X}$  with the directional derivative; after all,  $\overline{X}$ ’s also generate a copy of the universal enveloping algebra in  $\mathcal{W}(\mathfrak{g})$ . The answer is simple; the directional derivatives should commute with the  $\operatorname{Cl}(\mathfrak{g})$  factor.

With  $\widehat{X}$  identified as the directional derivative  $\partial_{\widetilde{X}}$ , Equation 5.2.22 then shows that the cubic Dirac operator  $\mathcal{D}$  is the geometric Dirac operator (Section 6.2.6) on a vector bundle over  $G$  associated to the connection  $\nabla$  defined by

$$\nabla_{\widetilde{X}} = \partial_{\widetilde{X}} + \frac{1}{3} c(\gamma(X)),$$

where  $c$  denotes the  $\operatorname{Cl}(\mathfrak{g})$ -module structure on the vector bundle. This is *not* a Clifford connection (Section 6.2.3), which demands that  $[\nabla_{\widetilde{X}}, c(Y)] = c(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})$  where  $\widetilde{\nabla}$  is the Riemannian connection on the

tangent bundle of  $G$ ; instead, we have

$$\begin{aligned} [\nabla_X, c(Y)] &= \frac{1}{3} [c(\gamma(X)), c(Y)] = \frac{1}{3} c([ \gamma(X), Y ]) \\ &= \frac{1}{3} c([X, Y]_{\mathfrak{g}}) = \frac{2}{3} c(\tilde{\nabla}_{\tilde{X}} \tilde{Y}), \end{aligned}$$

where the last equality follows from Equation 2.2.17.

### 5.3 THE CLASSICAL WEIL ALGEBRA

**5.3.1** Traditionally the Weil algebra of a (finite-dimensional) Lie algebra  $\mathfrak{g}$  is  $\wedge(\mathfrak{g})^* \otimes S(\mathfrak{g}^*)$ . (It has a  $\mathfrak{g}$ -differential algebra structure which we shall review shortly.) Since we assume that  $\mathfrak{g}$  is equipped with an invariant inner product, we shall identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$  by the inner product and, thus, the Weil algebra with  $\wedge(\mathfrak{g}) \otimes S(\mathfrak{g})$ .

The connection between the classical Weil algebra and the quantum Weil algebra is most obvious from Equation 5.2.20, as  $\wedge(\mathfrak{g})$  and  $S(\hat{\mathfrak{g}})$  are the associated graded algebras of  $\text{Cl}(\mathfrak{g})$  and  $\mathcal{U}(\hat{\mathfrak{g}})$ , respectively. To make this more precise, let  $\bigcup_{k \in \mathbb{Z}} \text{Cl}_k(\mathfrak{g})$  and  $\bigcup_{k \in \mathbb{Z}} \mathcal{U}_k(\hat{\mathfrak{g}})$  be the usual filtrations for  $\text{Cl}(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{g})$ , respectively (see Sections 4.1.26 and 5.1.10). Set

$$\text{Cl}_{(k)}(\mathfrak{g}) := \text{Cl}_k(\mathfrak{g}), \quad \mathcal{U}_{(2\ell)}(\hat{\mathfrak{g}}) := \mathcal{U}_{\ell}(\hat{\mathfrak{g}}).$$

Then, give  $\mathcal{W}(\mathfrak{g})$  the following filtration:

$$\mathcal{W}(\mathfrak{g}) = \bigcup_{q \in \mathbb{Z}} \mathcal{W}_q(\mathfrak{g})$$

where

$$\mathcal{W}_q(\mathfrak{g}) = \bigcup_{k+2\ell=q} \text{Cl}_{(k)}(\mathfrak{g}) \otimes \mathcal{U}_{(2\ell)}(\hat{\mathfrak{g}}).$$

The associated graded algebra of  $\mathcal{W}(\mathfrak{g})$ , which we denote by  $W(\mathfrak{g})$  instead of  $\text{gr } \mathcal{W}(\mathfrak{g})$  for reasons that will be obvious in a moment, is

$$W(\mathfrak{g}) = \wedge(\mathfrak{g}) \otimes S(\hat{\mathfrak{g}}).$$

The generators  $X$  in  $\mathfrak{g}$  and  $\hat{X}$  in  $\hat{\mathfrak{g}}$  are of degree 1 and 2.

Recall that the symbol maps for  $\text{Cl}(\mathfrak{g})$  and  $\mathcal{U}(\hat{\mathfrak{g}})$ , respectively, are given by the inverse of the Chevalley map 5.1.17 and the inverse of the Poincaré-Birkhoff-Witt isomorphism 4.1.12; their tensor product

$$q^{-1} \otimes \text{PBW}^{-1} : \mathcal{W}(\mathfrak{g}) = \text{Cl}(\mathfrak{g}) \otimes \mathcal{U}(\hat{\mathfrak{g}}) \rightarrow W(\mathfrak{g}) = \wedge(\mathfrak{g}) \otimes S(\hat{\mathfrak{g}}) \quad (5.3.2)$$

is the symbol map for the quantum Weil algebra. Through this symbol map, the  $\mathfrak{g}$ -differential algebra structure on  $\mathcal{W}(\mathfrak{g})$  induces a  $\mathfrak{g}$ -differential algebra structure on  $W(\mathfrak{g})$  as follows; we divide the procedure into two steps:

*Step 1:* Recall that the Lie derivatives on  $\mathcal{W}(\mathfrak{g})$  were defined in terms of the even generators  $\bar{X} = \hat{X} + \gamma(X)$ . We will need to define an analogous generator for  $W(\mathfrak{g})$ . The image of  $\bar{X}$  under the symbol

map is

$$q^{-1} \otimes \text{PBW}^{-1}(\widehat{X} + \gamma(X)) = \text{PBW}^{-1}(\widehat{X}) + q^{-1}\gamma(X) = \widehat{X} + \lambda(X).$$

Here  $\lambda(X)$  is the image of  $X$  under the map  $\lambda : \mathfrak{g} \rightarrow \wedge^2(\mathfrak{g})$  defined in Definition 5.1.46. With a slight abuse of notation, we write

$$\bar{X} := \widehat{X} + \lambda(X) \in W(\mathfrak{g}).$$

Clearly  $\bar{X}$  is of degree 2, and it may replace  $\widehat{X}$  as a generator for  $W(\mathfrak{g})$ . We denote the set of these new generators by

$$\bar{\mathfrak{g}} := \{\bar{X} \mid X \in \mathfrak{g}\} \subseteq W(\mathfrak{g}).$$

*Step 2:* Define the derivations  $\iota_X$ ,  $L_X$ , and  $d$  on  $W(\mathfrak{g})$  of degree  $-1$ ,  $0$ , and  $1$ , respectively, in such a way that the symbol map intertwines them with their counter parts on  $W(\mathfrak{g})$  at the level of generators. that means, if we apply the symbol map  $q^{-1} \otimes \text{PBW}^{-1}$  to Equations 5.2.6–5.2.8, we get (keeping in mind that  $q^{-1} \otimes \text{PBW}^{-1}$  maps the generating subspace  $\mathfrak{g} \oplus \bar{\mathfrak{g}}$  in  $W(\mathfrak{g})$  identically to  $\mathfrak{g} \oplus \bar{\mathfrak{g}}$  in  $W(\mathfrak{g})$ )

$$\iota_X Y = \langle X, Y \rangle, \quad \iota_X \bar{Y} = [X, Y]_{\mathfrak{g}}, \quad (5.3.3)$$

$$L_X Y = [X, Y]_{\mathfrak{g}}, \quad L_X \bar{Y} = \overline{[X, Y]_{\mathfrak{g}}}. \quad (5.3.4)$$

$$dX = \bar{X}, \quad d\bar{X} = 0. \quad (5.3.5)$$

With these derivations,  $W(\mathfrak{g})$  is exactly the classical Weil algebra (with the identification  $\mathfrak{g}^* \simeq \mathfrak{g}$ ), which is a  $\mathfrak{g}$ -differential algebra; see [3, § 3.2].

*Remark.* Let  $\bar{\mathfrak{g}}$  be the linear subspace spanned by the generators  $\bar{X} = \widehat{X} + \lambda(X)$  in  $W(\mathfrak{g})$ . Since  $\widehat{X}$  and  $\lambda(X)$  commutes with every generator in the generating subspace  $\mathfrak{g} \oplus \hat{\mathfrak{g}} \subseteq W(\mathfrak{g})$ , the new generators  $\bar{X}$  also commute with every generator in the generating subspace  $\mathfrak{g} \oplus \bar{\mathfrak{g}} \subseteq W(\mathfrak{g})$ . Thus, we have

$$W(\mathfrak{g}) = \wedge(\mathfrak{g}) \otimes S(\bar{\mathfrak{g}}).$$

**5.3.6 SUPER-SYMMETRIZATION.** Recall that the Chevalley map  $q : \wedge(\mathfrak{g}) \xrightarrow{\sim} \text{Cl}(\mathfrak{g})$  is the antisymmetrization map with respect to the generators in  $\mathfrak{g}$  (see Section 5.1.15) and that the Poincaré-Birkhoff-Witt isomorphism  $\text{PBW} : S(\hat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{U}(\hat{\mathfrak{g}})$  is the symmetrization map with respect to the generators in  $\hat{\mathfrak{g}}$  (see Section 4.1.9). Hence, the inverse of the symbol map 5.3.2 is the super-symmetrization map with respect to the generators  $X$  and  $\widehat{X}$ :

$$\begin{aligned} q \otimes \text{PBW} : W(\mathfrak{g}) = \wedge \mathfrak{g} \otimes S(\hat{\mathfrak{g}}) &\rightarrow W(\mathfrak{g}) = \text{Cl}(\mathfrak{g}) \otimes \mathcal{U}(\hat{\mathfrak{g}}), \\ v_1 \cdots v_k &\mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}_s(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)}. \end{aligned} \quad (5.3.7)$$

Here  $v_i$ 's are vectors in  $\mathfrak{g}$  or  $\hat{\mathfrak{g}}$ , and  $\text{sgn}_s(\sigma) = (-1)^N$  where  $N$  is the number of pairs of odd vectors  $v_i, v_j$  with  $i < j$  such that  $\sigma(j) < \sigma(i)$ . This map is a vector space isomorphism.

The super-symmetrization in terms of the generators  $X$  and  $\bar{X}$  is also possible. Following A. Alekseev and E. Meinrenken [2], we call this map the *quantization<sup>2</sup> map* and denote it by

$$\mathcal{Q} : W(\mathfrak{g}) = \wedge(\mathfrak{g}) \otimes S(\bar{\mathfrak{g}}) \rightarrow \mathcal{W}(\mathfrak{g}) = \text{Cl}(\mathfrak{g}) \rtimes \mathcal{U}(\bar{\mathfrak{g}}). \quad (5.3.8)$$

This is the tensor product of the Chevalley map on  $\wedge(\mathfrak{g})$  and the Poincaré-Birkhoff-Witt isomorphism on  $S(\bar{\mathfrak{g}})$ . This is also a vector space isomorphism.

*Remark.* Recall that the  $\mathfrak{g}$ -differential algebra structure for the classical Weil algebra was defined in terms of the generators  $X$  and  $\bar{X}$  (not  $\hat{X}$ ) so that it resembles the  $\mathfrak{g}$ -differential algebra structure of the quantum Weil algebra. As a consequence, the super-symmetrization with respect to the generators  $X$  and  $\bar{X}$ , namely the quantization map  $\mathcal{Q}$ , intertwines the derivations  $\iota_X$ ,  $L_X$ , and  $d$  as demonstrated in [3, Lem. 4.3, p. 315]. Now let  $D := \mathcal{Q}^{-1}(\mathcal{D})$  so that

$$d\mathcal{D} = d\mathcal{Q}(D) = \mathcal{Q}(dD).$$

Then, by Equation 5.2.9,

$$[\mathcal{D}, \mathcal{D}]_s = \mathcal{Q}(dD). \quad (5.3.9)$$

The left-hand side is equal to  $2\mathcal{D}^2$ . For  $dD$ , we have

$$dD = \sum_{i=1}^{\dim \mathfrak{g}} \hat{X}_i \hat{X}_i, \quad (5.3.10)$$

where  $\{\hat{X}_i\}_{i=1}^{\dim \mathfrak{g}}$  is any orthonormal basis for  $\mathfrak{g}$  [3, Prop. 5.4, p. 319]. Let us give the following notation for the element in the right-hand side of Equation 5.3.10:

$$\hat{\Delta}_{\mathfrak{g}} := \sum_{i=1}^{\dim \mathfrak{g}} \hat{X}_i \hat{X}_i. \quad (5.3.11)$$

<sup>2</sup> At this point the word “quantization” means the inverse of the symbol map of a filtered algebra. The meaning can be made richer with the notion of a super-Poisson bracket [66]; a *super-Poisson bracket* on a super-commutative algebra  $A$  (in other words, the super-commutator of any two elements of  $A$  is zero) is a map  $\{, \} : A \times A \rightarrow A$  such that, for homogeneous elements  $x, y$ , and  $z$ , we have (i)  $\{x, y\} = -(-1)^{\deg(x)\deg(y)}\{y, x\}$ , (ii)  $\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{\deg(x)\deg(y)}\{y, \{x, z\}\}$ , (iii) the map  $w \mapsto \{x, w\}$  defines a super-derivation of degree  $\deg(x)$ . As A. Alekseev and E. Meinrenken point out [3, Rmk.5.5(a), p. 319], one can define a super-Poisson bracket  $\{, \}$  on  $W(\mathfrak{g})$  in such a way that the quantization map  $\mathcal{Q}$  intertwines the super-Poisson bracket with the super-commutator at the level of generators (which resembles very much the principle of quantization laid out by P. A. M. Dirac [30, § 21]); all we need to do is to define  $\{, \}$  for the generators of  $W(\mathfrak{g})$  so that it resembles the super-commutator relations 5.2.2–5.2.4 in  $W(\mathfrak{g})$ ; that is, set  $\{X, Y\} = \langle X, Y \rangle$ ,  $\{\bar{X}, Y\} = [X, Y]_{\mathfrak{g}}$ , and  $\{\bar{X}, \bar{Y}\} = [\bar{X}, \bar{Y}]_{\mathfrak{g}}$ . Then the super-derivations  $\iota_X$ ,  $L_X$ , and  $d$  all become inner; in particular,  $d(\cdot) = \{\bar{D}, \cdot\}$  where  $D$  is the cubic element defined by Equation 5.3.14. This allows us to write Equation 5.3.9 in a more symmetric form:  $[\mathcal{D}, \mathcal{D}]_s = \mathcal{Q}\{D, D\}$ .

Then, Equation 5.3.9 can be written as

$$\mathcal{D}^2 = \frac{1}{2} \mathcal{Q}(\hat{\Delta}_{\mathfrak{g}}). \quad (5.3.12)$$

This has an important geometrical meaning; see Theorem 7.2.7.

The following lemma is from [3, Rmk.5.2(a), p. 317]; it gives an explicit expression for  $\mathcal{D}$ :

**5.3.13 LEMMA.** *Let  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  be any orthonormal basis for  $\mathfrak{g}$ . Let  $\mathcal{D} := \mathcal{Q}^{-1}(\mathcal{D})$ . Then*

$$\mathcal{D} = \sum_{i=1}^{\dim \mathfrak{g}} X_i \left( \bar{X}_i - \frac{2}{3} \lambda(X_i) \right), \quad (5.3.14)$$

where  $\lambda(X_i)$  is the image of  $X_i$  under the map  $\lambda$  defined in Definition 5.1.46.

*Proof.* We shall verify that the right-hand side of Equation 5.3.14 is indeed mapped to  $\mathcal{D}$  under the quantization map. In other words, we wish to show that

$$\mathcal{Q} \left( \sum_{i=1}^{\dim \mathfrak{g}} X_i \bar{X}_i \right) - \frac{2}{3} \mathcal{Q} \left( \sum_{i=1}^{\dim \mathfrak{g}} X_i \lambda(X_i) \right) = \mathcal{D}. \quad (5.3.15)$$

The quantization map is the super-symmetrization with respect to the generators  $X$  and  $\bar{X}$ . Thus,

$$\mathcal{Q}(X_i \bar{X}_i) = \frac{1}{2} (X_i \bar{X}_i + \bar{X}_i X_i).$$

Since  $[\bar{X}_i, X_i]_s = [X_i, X_i]_g = 0$ , we have  $\bar{X}_i X_i = X_i \bar{X}_i$ . So

$$\mathcal{Q}(X_i \bar{X}_i) = X_i \bar{X}_i. \quad (5.3.16)$$

Next, we calculate  $\mathcal{Q}(\sum_{i=1}^{\dim \mathfrak{g}} X_i \lambda(X_i))$ . By Equation 5.1.53, we have

$$-2 \sum_{i=1}^{\dim \mathfrak{g}} X_i \lambda(X_i) = \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_g \rangle X_i X_j X_k. \quad (5.3.17)$$

Since the above element is in the exterior algebra  $\wedge(\mathfrak{g}) \otimes \{1\}$  in  $W(\mathfrak{g})$ , the quantization map operates on it as the antisymmetrization map. So

$$\begin{aligned} \mathcal{Q} \left( \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_g \rangle X_i X_j X_k \right) \\ = \frac{1}{3!} \sum_{\sigma \in S_3} \text{sgn}(\sigma) \left( \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_g \rangle X_{\sigma(i)} X_{\sigma(j)} X_{\sigma(k)} \right). \end{aligned}$$

But, since  $\text{sgn}(\sigma)X_{\sigma(i)}X_{\sigma(j)}X_{\sigma(k)} = X_iX_jX_k$ ,

$$\begin{aligned} \mathcal{Q}\left(\sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_{\mathfrak{g}} \rangle X_iX_jX_k\right) \\ = \frac{1}{3!} \sum_{\sigma \in S_3} \left( \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_{\mathfrak{g}} \rangle X_iX_jX_k \right) \\ = \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_{\mathfrak{g}} \rangle X_iX_jX_k. \end{aligned}$$

This proves, by Equation 5.3.17, that

$$\mathcal{Q}\left(-2 \sum_{i=1}^{\dim \mathfrak{g}} X_i \lambda(X_i)\right) = \sum_{i,j,k=1}^{\dim \mathfrak{g}} \langle X_i, [X_j, X_k]_{\mathfrak{g}} \rangle X_iX_jX_k.$$

Then, by Equation 5.1.55,

$$\mathcal{Q}\left(-2 \sum_{i=1}^{\dim \mathfrak{g}} X_i \lambda(X_i)\right) = -2 \sum_{i=1}^{\dim \mathfrak{g}} X_i \gamma(X_i). \quad (5.3.18)$$

Collecting Equations 5.2.10, 5.3.16 and 5.3.18, we have

$$\mathcal{Q}\left(\sum_{i=1}^{\dim \mathfrak{g}} X_i \bar{X}_i\right) - \frac{2}{3} \mathcal{Q}\left(\sum_{i=1}^{\dim \mathfrak{g}} X_i \lambda(X_i)\right) = \sum_{i=1}^{\dim \mathfrak{g}} X_i \bar{X}_i - \frac{2}{3} \sum_{i=1}^{\dim \mathfrak{g}} X_i \gamma(X_i) = \mathcal{P}.$$

This is our desired result Equation 5.3.15.  $\square$

**5.3.19 DUFLO ISOMORPHISM REVISITED.** In their paper [3, Thm. 5.3, p. 318], A. Alekseev and E. Meinrenken compared the two supersymmetrization maps  $q \otimes \text{PBW}$  and  $\mathcal{Q}$ , and found that

$$\mathcal{Q} = (q \otimes \text{PBW}) \circ \partial_S, \quad (5.3.20)$$

where  $\partial_S$  is an infinite order differential operator with coefficients in  $\wedge(\mathfrak{g})$  that is defined as follows. Take the power series of

$$\mathcal{S}(X) = j(X) \otimes e^{\lambda\phi(X)} \quad (5.3.21)$$

where

$$j(X) = \det^{1/2} \left( \frac{\sinh(\text{ad}_X/2)}{\text{ad}_X/2} \right), \quad \phi(X) = 2 \coth\left(\frac{\text{ad}_X}{2}\right) - \frac{4}{\text{ad}_X}. \quad (5.3.22)$$

View  $\mathcal{S}(X)$  as an element of  $\wedge(\mathfrak{g}) \otimes S(\mathfrak{g})^*$  using the isomorphism  $\mathfrak{g}^* \simeq \mathfrak{g}$  provided by the inner product. Then  $\partial_S$  is the operator we get by having the  $S(\mathfrak{g})^*$  factor act on  $\{1\} \otimes S(\hat{\mathfrak{g}})$  as an infinite order differential operator (here we use the isomorphism  $\hat{\mathfrak{g}} \simeq \mathfrak{g}$ ) and having the  $\wedge(\mathfrak{g})$  factor act on  $\wedge(\mathfrak{g}) \otimes \{1\}$  by contraction (obtained by extending the contraction map  $\iota : \mathfrak{g} \rightarrow \text{End}(\wedge(\mathfrak{g}))$  to  $\wedge(\mathfrak{g}) \rightarrow \text{End}(\wedge(\mathfrak{g}))$  as an algebra homomorphism). Notice the appearance of the function  $j$ ,

which is also present in the formulation of the Duflo isomorphism. A. Alekseev and E. Meinrenken used Equation 5.3.20 to prove the Duflo isomorphism for finite-dimensional Lie algebras over  $\mathbb{R}$  equipped with an inner product that is skew-invariant under the  $\text{ad}(\mathfrak{g})$ -action. In what follows is a sketch of their proof.

A few definitions are in order. Let  $(A, \iota, L, d)$  be a  $\mathfrak{g}$ -differential algebra. An element  $a$  of  $A$  is said to be *horizontal* if  $\iota_X a = 0$  for all  $X$  in  $\mathfrak{g}$ ; it is said to be *invariant* if  $L_X a = 0$  for all  $X$  in  $\mathfrak{g}$ . The horizontal and the invariant elements, respectively, constitute the *horizontal subalgebra*  $A_{\text{hor}}$  and the *invariant subalgebra*  $A^{\mathfrak{g}}$  of  $A$ . The *basic* elements of  $A$  are the ones that are both horizontal and invariant; they constitute the *basic subalgebra*  $A_{\text{bas}} := A_{\text{hor}} \cap A^{\mathfrak{g}}$  of  $A$ .

From the commutator relations 5.2.13–5.2.15, we see that the horizontal subalgebra of  $\mathcal{W}(\mathfrak{g})$  is  $\mathcal{U}(\hat{\mathfrak{g}})$ . So the basic subalgebra of  $\mathcal{W}(\mathfrak{g})$  is the invariant subalgebra of  $\mathcal{U}(\hat{\mathfrak{g}})$ . Now, by definition,  $L_X \hat{Y} = [\bar{X}, \hat{Y}]_s$ . Using the commutator relations, we find that  $[\bar{X}, \hat{Y}]_s = \widehat{[\bar{X}, Y]_{\mathfrak{g}}}$ . Thus, the invariant subalgebra of  $\mathcal{U}(\hat{\mathfrak{g}})$  must be the  $\text{ad}(\hat{\mathfrak{g}})$ -invariant part of  $\mathcal{U}(\hat{\mathfrak{g}})$ , namely, the center of  $\mathcal{U}(\hat{\mathfrak{g}})$ . Therefore,  $\mathcal{W}(\mathfrak{g})_{\text{bas}} = \mathcal{Z}(\hat{\mathfrak{g}})$ . By the same token, we have  $\mathcal{W}(\mathfrak{g}) = S(\hat{\mathfrak{g}})^{\mathfrak{g}}$ , where  $S(\hat{\mathfrak{g}})^{\mathfrak{g}}$  is the  $\text{ad}(\hat{\mathfrak{g}})$ -invariant part of  $S(\hat{\mathfrak{g}})$ .

Because the quantization map  $\mathcal{Q} : \mathcal{W}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{W}(\mathfrak{g})$  intertwines the contractions and the Lie derivatives, its restriction to the basic subalgebra gives a vector space isomorphism

$$\mathcal{Q}^{\mathfrak{g}} : S(\hat{\mathfrak{g}})^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{Z}(\hat{\mathfrak{g}}). \quad (5.3.23)$$

The quantization map also intertwines the differential; so the above map induces a linear isomorphism on the cohomologies,

$$\mathcal{Q}^{\mathfrak{g}*} : H^*\{S(\hat{\mathfrak{g}})^{\mathfrak{g}}\} \rightarrow H^*\{\mathcal{Z}(\hat{\mathfrak{g}})\}.$$

This turns out to be an *algebra* isomorphism [3, Thm. 4.7, p. 316]. But notice that  $S(\hat{\mathfrak{g}})^{\mathfrak{g}}$  and  $\mathcal{Z}(\hat{\mathfrak{g}})$ , respectively, are in the even subspace of  $\mathcal{W}(\mathfrak{g})$  and  $\mathcal{W}(\mathfrak{g})$ . Hence,  $H^*\{S(\hat{\mathfrak{g}})^{\mathfrak{g}}\} = S(\hat{\mathfrak{g}})^{\mathfrak{g}}$ ,  $H^*\{\mathcal{Z}(\hat{\mathfrak{g}})\} = \mathcal{Z}(\hat{\mathfrak{g}})$ , and  $\mathcal{Q}^{\mathfrak{g}*} = \mathcal{Q}^{\mathfrak{g}}$ . Therefore, the vector space isomorphism 5.3.23 is an algebra isomorphism.

Finally, since the elements in  $S(\hat{\mathfrak{g}})^{\mathfrak{g}}$  have no exterior algebra factor, to the end of applying the operator 5.3.20 on  $S(\hat{\mathfrak{g}})^{\mathfrak{g}}$ , we may drop the exterior algebra factor in the power series 5.3.21 and replace  $\mathcal{S}(X)$  with  $j(X)$ . To sum up, the algebra isomorphism 5.3.23 is equal to  $(q \otimes \text{PBW}) \circ \partial_j$ . This is precisely the Duflo isomorphism.

## 5.4 THE RELATIVE WEIL ALGEBRA

**5.4.1 DEFINITION.** Let  $(A, \iota, L, d)$  be a  $\mathfrak{g}$ -differential algebra. Let  $\mathfrak{k}$  be a subset of  $\mathfrak{g}$ . An element  $a$  of  $A$  is said to be  $\mathfrak{k}$ -horizontal if  $\iota_X a = 0$  for all  $X$  in  $\mathfrak{k}$ ; it is said to be  $\mathfrak{k}$ -invariant if  $L_X a = 0$  for all  $X$  in  $\mathfrak{k}$ . The  $\mathfrak{k}$ -horizontal and the  $\mathfrak{k}$ -invariant elements, respectively, constitute the  $\mathfrak{k}$ -horizontal subalgebra  $A_{\mathfrak{k}\text{-hor}}$  and the  $\mathfrak{k}$ -invariant subalgebra  $A^{\mathfrak{k}}$  of



A. The  $\mathfrak{k}$ -basic elements of  $A$  are the ones that are both horizontal and invariant with respect to  $\mathfrak{k}$ ; they constitute the  $\mathfrak{k}$ -basic subalgebra  $A_{\mathfrak{k}\text{-bas}} := A_{\mathfrak{k}\text{-hor}} \cap A^{\mathfrak{k}}$  of  $A$ . In the special case where  $\mathfrak{k} = \mathfrak{g}$ , we shall omit the qualifier  $\mathfrak{k}$  from the above terminologies.

5.4.2 Let  $\mathfrak{k}$  be any Lie subalgebra of  $\mathfrak{g}$ , so that

$$[\mathfrak{k}, \mathfrak{k}]_{\mathfrak{g}} \subseteq \mathfrak{k}. \quad (5.4.3)$$

Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be an orthogonal decomposition. Owing to the invariance of the inner product, we have

$$\langle [\mathfrak{k}, \mathfrak{p}]_{\mathfrak{g}}, \mathfrak{k} \rangle = -\langle \mathfrak{p}, [\mathfrak{k}, \mathfrak{k}]_{\mathfrak{g}} \rangle = 0.$$

Hence,

$$[\mathfrak{k}, \mathfrak{p}]_{\mathfrak{g}} \subseteq \mathfrak{p}. \quad (5.4.4)$$

5.4.5 Following A. Alekseev and E. Meinrenken [3], we call the  $\mathfrak{k}$ -basic subalgebra of  $\mathcal{W}(\mathfrak{g})$  as the *relative (quantum) Weil algebra* for the pair  $(\mathfrak{g}, \mathfrak{k})$  and denote it by  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$ . The contraction  $\iota_X$  on  $\mathcal{W}(\mathfrak{g})$  is the inner derivation with respect to  $X$ ; so the super-commutator relations  $[X, Y]_s = [X, Y]_{\mathfrak{g}}$  and  $[X, \hat{Y}]_s = 0$  tells us that the  $\mathfrak{k}$ -horizontal subalgebra is

$$\mathcal{W}(\mathfrak{g})_{\mathfrak{k}\text{-hor}} = \text{Cl}(\mathfrak{p}) \otimes \mathcal{U}(\hat{\mathfrak{g}}).$$

Hence,

$$\mathcal{W}(\mathfrak{g}, \mathfrak{k}) = (\text{Cl}(\mathfrak{p}) \otimes \mathcal{U}(\hat{\mathfrak{g}}))^{\mathfrak{k}}.$$

The differential on  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  inherited from  $\mathcal{W}(\mathfrak{g})$  is again inner [3, Prop. 6.4, p. 321] with respect to the element

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k}) := \mathcal{D}_{\mathfrak{g}} - \mathcal{D}_{\mathfrak{k}}, \quad (5.4.6)$$

where  $\mathcal{D}_{\mathfrak{g}}$  denotes the cubic Dirac operator in  $\mathcal{W}(\mathfrak{g})$  and  $\mathcal{D}_{\mathfrak{k}}$  denotes the image of the cubic Dirac operator in  $\mathcal{W}(\mathfrak{k})$  under the canonical inclusion  $\mathcal{W}(\mathfrak{k}) \hookrightarrow \mathcal{W}(\mathfrak{g})$  induced by the generators  $X$  and  $\bar{X}$ . The element  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  is called the *relative Dirac operator* for the pair  $(\mathfrak{g}, \mathfrak{k})$ .

5.4.7 DEFINITION. We denote by

$$\begin{aligned} \gamma^{\mathfrak{g}} : \mathfrak{g} &\rightarrow \mathfrak{spin}(\mathfrak{g}) \subseteq \text{Cl}(\mathfrak{g}), \\ \gamma^{\mathfrak{k}} : \mathfrak{k} &\rightarrow \mathfrak{spin}(\mathfrak{k}) \subseteq \text{Cl}(\mathfrak{k}), \end{aligned}$$

the Lie algebra homomorphisms constructed in Definition 5.1.46. We denote by

$$\gamma^{\mathfrak{p}} : \mathfrak{k} \rightarrow \mathfrak{spin}(\mathfrak{p}) \subseteq \text{Cl}(\mathfrak{p})$$

the Lie algebra homomorphism obtained by composing  $\mathfrak{k} \xrightarrow{\text{ad}} \mathfrak{so}(\mathfrak{p})$  with the Lie algebra isomorphism  $\mathfrak{so}(\mathfrak{p}) \rightarrow \mathfrak{spin}(\mathfrak{p})$  given by the map 5.1.42.

*Remark.* Let  $\{X_i\}_{i=1}^{\dim \mathfrak{k}}$  and  $\{X^i\}_{i=1}^{\dim \mathfrak{k}}$  be two basis for  $\mathfrak{k}$  that are dual with respect to the inner product so that  $\langle X_i, X^j \rangle = \delta_{ij}$ . Likewise, let  $\{Y_j\}_{j=1}^{\dim \mathfrak{p}}$  be a basis for  $\mathfrak{p}$  and let  $\{Y^j\}_{j=1}^{\dim \mathfrak{p}}$  be the dual basis. Let  $\{Z_i\}_{i=1}^{\dim \mathfrak{g}} = \{X_i\}_{i=1}^{\dim \mathfrak{k}} \sqcup \{Y_j\}_{j=1}^{\dim \mathfrak{p}}$ ; define  $\{Z^i\}_{i=1}^{\dim \mathfrak{g}}$  similarly. By Equation 5.1.55, we have, for  $X$  in  $\mathfrak{k}$ ,

$$\gamma^{\mathfrak{g}}(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{g}} \langle X, [Z_i, Z_j]_{\mathfrak{g}} \rangle Z^i Z^j, \quad (5.4.8)$$

$$\gamma^{\mathfrak{k}}(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{k}} \langle X, [X_i, X_j]_{\mathfrak{g}} \rangle X^i X^j, \quad (5.4.9)$$

$$\gamma^{\mathfrak{p}}(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{p}} \langle X, [Y_i, Y_j]_{\mathfrak{g}} \rangle Y^i Y^j. \quad (5.4.10)$$

**5.4.11 LEMMA.** *Let  $\{X_i\}_{i=1}^{\dim \mathfrak{k}}$  and  $\{Y_j\}_{j=1}^{\dim \mathfrak{p}}$  be orthonormal basis for  $\mathfrak{k}$  and  $\mathfrak{p}$ , respectively; and let  $\{Z_i\}_{i=1}^{\dim \mathfrak{g}} = \{X_i\}_{i=1}^{\dim \mathfrak{k}} \sqcup \{Y_j\}_{j=1}^{\dim \mathfrak{p}}$ . In the following, let  $X$  be an arbitrary vector in  $\mathfrak{k}$ .*

(a) *We have*

$$\gamma^{\mathfrak{g}}(X) = \gamma^{\mathfrak{k}}(X) + \gamma^{\mathfrak{p}}(X). \quad (5.4.12)$$

(b) *Under the canonical inclusion  $\mathcal{W}(\mathfrak{k}) \hookrightarrow \mathcal{W}(\mathfrak{g})$ , we have*

$$\widehat{X} \mapsto \widehat{X} + \gamma^{\mathfrak{p}}(X). \quad (5.4.13)$$

*Proof.* (a) Let us rewrite Equation 5.4.8 as follows:

$$\begin{aligned} \gamma^{\mathfrak{g}}(X) = & \underbrace{\frac{1}{2} \sum_{i=1}^{\dim \mathfrak{k}} \sum_{j=i}^{\dim \mathfrak{k}} \langle \text{ad}_X X_i, X_j \rangle X_i X_j}_{\textcircled{1}} \\ & + \underbrace{\sum_{i=1}^{\dim \mathfrak{k}} \sum_{j=1}^{\dim \mathfrak{p}} \langle \text{ad}_X X_i, Y_j \rangle X_i Y_j}_{\textcircled{2}} \\ & + \underbrace{\frac{1}{2} \sum_{i=1}^{\dim \mathfrak{p}} \sum_{j=i}^{\dim \mathfrak{p}} \langle \text{ad}_X Y_i, Y_j \rangle Y_i Y_j}_{\textcircled{3}}. \end{aligned} \quad (5.4.14)$$

Now, owing to the invariance of the inner product, the orthogonal decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is invariant under  $\text{ad}(\mathfrak{k})$ -action. As a consequence, the sums  $\textcircled{1}$ ,  $\textcircled{2}$ , and  $\textcircled{3}$  above are equal to  $\gamma^{\mathfrak{k}}(X)$ , 0, and  $\gamma^{\mathfrak{p}}(X)$ , respectively.

(b) The generator  $\widehat{X}$  in  $\mathcal{W}(\mathfrak{k})$  is, by definition, equal to  $\bar{X} - \gamma^{\mathfrak{k}}(X)$ . So, under the canonical inclusion  $\mathcal{W}(\mathfrak{k}) \hookrightarrow \mathcal{W}(\mathfrak{g})$ ,

$$\widehat{X} = \bar{X} - \gamma^{\mathfrak{k}}(X) \mapsto \bar{X} - \gamma^{\mathfrak{k}}(X).$$

Since  $\hat{X} = \bar{X} - \gamma^g(X)$  in  $\mathcal{W}(\mathfrak{g})$ , the image above is equal to  $\hat{X} + \gamma^g(X) - \gamma^t(X)$ . By Equation 5.4.12, this is equal to  $\hat{X} + \gamma^p(X)$ .  $\square$

5.4.15 NOTATION. The restriction of the canonical inclusion 5.4.13 onto  $\{1\} \otimes \mathcal{U}(\hat{\mathfrak{k}})$  in  $\mathcal{W}(\mathfrak{k})$  gives us a diagonal embedding of  $\mathcal{U}(\hat{\mathfrak{k}})$  into  $\mathcal{W}(\mathfrak{g})$ . We shall denote this embedding by

$$\begin{aligned} \text{diag}_{\mathcal{W}} : \mathcal{U}(\hat{\mathfrak{k}}) &\hookrightarrow \mathcal{W}(\mathfrak{g}, \mathfrak{k}), \\ \hat{X} &\mapsto \hat{X} + \gamma^p(X). \end{aligned} \quad (5.4.16)$$

5.4.17 The significance of the relative Dirac operator comes from the following result of B. Kostant [65, Thm. 2.13, p. 474]:

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2 = \frac{1}{2} \hat{\Omega}_{\mathfrak{g}} - \frac{1}{2} \text{diag}_{\mathcal{W}} \hat{\Omega}_{\mathfrak{k}} + \frac{1}{48} \text{tr}_{\mathfrak{g}} \hat{\Omega}_{\mathfrak{g}} - \frac{1}{48} \text{tr}_{\mathfrak{k}} \hat{\Omega}_{\mathfrak{k}}. \quad (5.4.18)$$

Here  $\hat{\Omega}_{\mathfrak{g}}$  denotes the Casimir element in  $\mathcal{U}(\hat{\mathfrak{g}})$ , and  $\text{tr}_{\mathfrak{g}} \hat{\Omega}_{\mathfrak{g}}$  denotes the trace of its representation on  $\hat{\mathfrak{g}}$  via the adjoint representation. In terms of an orthonormal basis  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  for  $\mathfrak{g}$ , we have  $\hat{\Omega}_{\mathfrak{g}} = \sum_{i=1}^{\dim \mathfrak{g}} \hat{X}_i \hat{X}_i$  (Equation 2.2.16). We used the caret symbol so that  $\hat{\Omega}_{\mathfrak{g}}$  would not be confused with the element  $\sum_{i=1}^{\dim \mathfrak{g}} \bar{X}_i \bar{X}_i$  in  $\{1\} \otimes \mathcal{U}(\bar{\mathfrak{g}})$  in  $\mathcal{W}(\mathfrak{g})$ . B. Kostant also demonstrated a simple formula [65, Eq. 1.85] for the trace of the Casimir element that holds when the Lie algebra  $\mathfrak{g}$  is compact (or reductive):

$$\frac{1}{24} \text{tr}_{\mathfrak{g}} \hat{\Omega}_{\mathfrak{g}} = -\|\rho_{\mathfrak{g}}\|^2. \quad (5.4.19)$$

Here,  $\rho_{\mathfrak{g}}$  is half the sum of the positive roots of  $\mathfrak{g}$ , and  $\|\rho_{\mathfrak{g}}\|$  is its norm induced by the inner product on  $\mathfrak{g}$ . This formula generalizes the “strange formula” of H. Freudenthal and H. de Vries [42, pp. 224 and 243] for the special case where  $\mathfrak{g}$  is semisimple and the inner product is coming from the Killing form.

5.4.20 What we have stated so far have their counter part in the classical Weil algebra. The classical relative Weil algebra  $W(\mathfrak{g}, \mathfrak{k})$  is defined to be the  $\mathfrak{k}$ -basic subalgebra of  $W(\mathfrak{g})$ . Similar argument used for  $W(\mathfrak{g}, \mathfrak{k})$  shows that

$$W(\mathfrak{g}, \mathfrak{k}) = (\wedge(\mathfrak{p}) \otimes S(\hat{\mathfrak{g}}))^{\mathfrak{k}}.$$

We set

$$D(\mathfrak{g}, \mathfrak{k}) := D_{\mathfrak{g}} - D_{\mathfrak{k}}, \quad (5.4.21)$$

where  $D_{\mathfrak{g}}$  is the cubic element 5.3.14 for  $W(\mathfrak{g})$ , and  $D_{\mathfrak{k}}$  is the image of the cubic element in  $W(\mathfrak{k})$  under the canonical inclusion  $W(\mathfrak{k}) \hookrightarrow W(\mathfrak{g})$  in terms of the generators  $X$  and  $\bar{X}$ .

The canonical inclusion  $W(\mathfrak{k}) \hookrightarrow W(\mathfrak{g})$  induces the diagonal embedding

$$\begin{aligned} \text{diag}_{\mathcal{W}} : S\hat{\mathfrak{k}} &\hookrightarrow W(\mathfrak{g}, \mathfrak{k}), \\ \hat{X} &\mapsto \hat{X} + \lambda^p(X), \end{aligned} \quad (5.4.22)$$

where  $\lambda^p$  is the composition of the map  $\gamma^p$  with the symbol map  $\text{Cl}(\mathfrak{p}) \rightarrow \wedge^p$ .

By Equations 5.3.12 and 5.4.18, we have

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2 = \frac{1}{2} \mathcal{Q}(\hat{\Delta}_{\mathfrak{g}} - \text{diag}_W \hat{\Delta}_{\mathfrak{k}}). \quad (5.4.23)$$

*Remark (Vogan Conjecture).* Because the quantization map  $\mathcal{Q} : W(\mathfrak{g}) \rightarrow \mathcal{W}(\mathfrak{g})$  intertwines the derivations  $\iota_X$ ,  $L_X$ , and  $d$ , the restriction of  $\mathcal{Q}$  to the  $\mathfrak{k}$ -basic subalgebra gives a vector space isomorphism

$$\mathcal{Q}^{\mathfrak{k}} : W(\mathfrak{g}, \mathfrak{k}) \xrightarrow{\sim} \mathcal{W}(\mathfrak{g}, \mathfrak{k}).$$

The induced map on the cohomologies,

$$\mathcal{Q}^{\mathfrak{k}*} : H^*\{W(\mathfrak{g}, \mathfrak{k})\} \rightarrow H^*\{\mathcal{W}(\mathfrak{g}, \mathfrak{k})\},$$

is also a vector space isomorphism. A. Alekseev and E. Meinrenken showed that this is in fact an *algebra* isomorphism [3, Thm. 6.5, p. 322], and used it to prove the Vogan conjecture.

In the special case where  $\mathfrak{k} = \mathfrak{g}$ , we have

$$W(\mathfrak{g}, \mathfrak{g}) = S(\hat{\mathfrak{g}})^{\mathfrak{g}}, \quad \mathcal{W}(\mathfrak{g}, \mathfrak{g}) = (\mathcal{U}(\hat{\mathfrak{g}}))^{\mathfrak{g}} = \mathcal{Z}(\hat{\mathfrak{g}}).$$

This is the situation we examined in Section 5.3.19, where we saw that the induced map  $\mathcal{Q}^{\mathfrak{g}*}$  is the Duflo isomorphism.

# 6

## EQUIVARIANT DIFFERENTIAL OPERATORS AND THE QUANTUM WEIL ALGEBRA

OUR goal of this chapter is to show that the relative Weil algebra  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  serves as an algebraic model for the space of certain  $G$ -equivariant differential operators on some vector bundle over the homogeneous space  $G/K$ , where  $G$  is a connected Lie group endowed with a bi-invariant metric and  $K$  a closed connected subgroup of  $G$ . We have hinted in a previous remark (see page 81) how this ought to be carried out.

We begin by reviewing some basic notions surrounding principal bundles and associated vector bundles. Clifford module bundles and spin structures are discussed next. We then describe a natural procedure for constructing a  $G$ -equivariant Clifford module bundle over  $G/K$ , provided that  $G/K$  has spin structure. Finally, we show how an element of the relative Weil algebra can be identified as a  $G$ -equivariant differential operator acting on the sections of an equivariant Clifford module bundle. For background materials and general reference, we refer to [85, Ch.2–4].

### 6.1 PRINCIPAL BUNDLES AND ASSOCIATED VECTOR BUNDLES

6.1.1 Let  $G$  be any Lie group. Let  $P$  be a principal  $G$ -bundle over a manifold  $M$ , and assume that  $G$  acts on  $P$  on the right. For any left  $G$ -vector space  $E$ , the *Borel mixing space*  $P \times_G E$  is the orbit space of the  $G$ -action on  $P \times E$  given by  $(p, v) \cdot g = (p \cdot g, g^{-1} \cdot v)$  for  $g$  in  $G$  and  $(p, v)$  in  $P \times E$ .

If  $E$  comes from a finite-dimensional representation  $\nu : G \rightarrow \text{Aut}(E)$ , then the mixing space  $P \times_G E$  is a vector bundle over  $M$

whose fibers are isomorphic to  $E$  (see [63, Prop. 5.4, p. 55]). This bundle is called the *vector bundle associated to  $P$*  by the representation  $\nu$ . If there is a need to specify the representation, then we shall write  $P \times_\nu E$  for  $P \times_G E$ .

**6.1.2 NOTATION.** Let  $P$  and  $E$  be as in Section 6.1.1. For  $(p, \nu)$  in  $P \times E$ , we shall denote its  $G$ -orbit by  $[p, \nu]$ . Thus,

$$[p \cdot g, g^{-1} \cdot \nu] = [p, \nu]$$

for all  $g$  in  $G$ .

*Example (Frame Bundles).* Let  $F \rightarrow M$  be a vector bundle of rank  $n$  over  $\mathbb{R}$ . An ordered basis  $(e_1, \dots, e_n)$  for the fiber  $F_x$  is called a *frame*. Let  $\text{Fr}(F)$  be the set of all frames of  $F$ . There is a canonical projection  $\text{Fr}(F) \rightarrow M$ , whose fiber  $\text{Fr}(F)_x$  over  $x$  in  $M$  is the set of all frames for  $F_x$ . Since the general linear group  $\text{GL}(n, \mathbb{R})$  acts freely and transitively on each fiber,  $\text{Fr}(F)$  is a principal  $\text{GL}(n, \mathbb{R})$ -bundle. (For details on the local trivializations, see [63, Ex. 5.2, p. 55].)

With the canonical action of  $\text{GL}(n, \mathbb{R})$  on  $\mathbb{R}^n$ , we can construct the associated vector bundle  $\text{Fr}(F) \times_{\text{GL}(n, \mathbb{R})} \mathbb{R}^n$  over  $M$ . A point in the fiber  $\text{Fr}(F) \times_{\text{GL}(n, \mathbb{R})} \mathbb{R}^n$  over  $x$  in  $M$  is an equivalence class  $[f, \nu]$  where  $f$  is a frame  $(e_1, \dots, e_n)$  for  $F_x$  and  $\nu$  is an  $n$ -tuple  $(\nu_1, \dots, \nu_n)$  in  $\mathbb{R}^n$ . The mapping  $[f, \nu] \mapsto \sum_{i=1}^n \nu_i e_i$  gives a well-defined bundle isomorphism  $\text{Fr}(F) \times_{\text{GL}(n, \mathbb{R})} \mathbb{R}^n \xrightarrow{\sim} F$ . Thus, every vector bundle is an associated vector bundle to a principal bundle.

If the bundle  $F$  is Riemannian (that is, an inner product is assigned for each fiber in a smooth manner) then we can construct the principal  $O(n)$ -bundle  $\text{Fr}_O(F)$  of orthonormal frames of  $F$ . If the Riemannian bundle  $F$  is orientable, then we can construct the principal  $SO(n)$ -bundle  $\text{Fr}_{SO}(F)$  of oriented orthonormal frames of  $F$ .

*Remark.* A smooth section of  $\text{Fr}(F)$  is called a *framing* of  $F$ . Not all vector bundles admit a global framing; however, every vector bundle admits local framings.

**6.1.3 CONNECTION.** For  $X$  in  $\mathfrak{g}$ , we denote by  $\tilde{X}$  the vector field on  $P$  whose value at  $p$  in  $P$  is defined by

$$\tilde{X}_p f = \left. \frac{d}{dt} \right|_0 f(p \cdot \exp(tX)), \quad (6.1.4)$$

where  $f$  is any smooth functions on  $P$ . This vector field is called the *fundamental vector field* on  $P$  generated by  $X$ . A tangent vector  $Y$  at  $p$  in  $P$  is said to be *vertical* if there is some  $X$  in  $\mathfrak{g}$  such that  $\tilde{X}_p = Y$ ; the vertical vectors at  $p$  span the *vertical subspace*  $V_p P$  of the tangent space  $T_p P$ . A vector field on  $P$  is said to be *vertical* if it is a section of the subbundle  $VP := \bigsqcup_{p \in P} V_p P$  of the tangent bundle  $TP$ .

A *connection* on a principal  $G$ -bundle  $P$  is a  $\mathfrak{g}$ -valued 1-form  $\theta$  on

$P$  (that is, an element of  $\Omega^1(P) \otimes \mathfrak{g}$ ), such that

$$\theta(\tilde{X}) = X, \quad \forall X \in \mathfrak{g}, \quad (6.1.5)$$

$$\text{Ad}_g \circ \theta = r_{g^{-1}}^* \theta, \quad \forall g \in G, \quad (6.1.6)$$

where  $r_{g^{-1}}^*$  denotes the pullback along the right-action of  $g$  on  $P$ , and  $\text{Ad}_g$  acts on the  $\mathfrak{g}$ -factor of the  $\mathfrak{g}$ -valued form. In regard to the properties 6.1.5 and 6.1.6, we say that the connection is vertical and invariant (or equivariant), respectively.

A connection  $\theta$  determines, at each point  $p$  in  $P$ , a complementary subspace  $H_p P$  of the vertical subspace  $V_p P$  in  $T_p P$ , namely,

$$H_p P := \ker(\theta_p).$$

This subspace is called the *horizontal subspace* at  $p$  with respect to  $\theta$ . Tangent vectors in  $H_p P$  are said to be *horizontal*, and so are the sections of the subbundle  $HP := \bigsqcup_{p \in P} H_p P$  of  $TP$ .

*Example (Maurer-Cartan Connection on Lie Groups).* Let  $P = G$  and view it as the principal  $G$ -bundle  $G \rightarrow \text{pt}$ . Let  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  be a basis for  $\mathfrak{g}$ . Let  $\{\alpha_i\}_{i=1}^{\dim \mathfrak{g}}$  be the dual basis for  $\mathfrak{g}^*$ . Then

$$\theta_e := \sum_{i=1}^{\dim \mathfrak{g}} \alpha_i \otimes X_i$$

defines a  $\mathfrak{g}$ -valued form on the tangent space  $T_e G$ . Extend this as a  $\mathfrak{g}$ -valued differential form  $\theta$  on  $P$  by defining the value of  $\theta$  at  $g \in G$  as

$$\theta_g = \text{Ad}_{g^{-1}} \circ r_{g^{-1}}^* \theta_e = \sum_{i=1}^{\dim \mathfrak{g}} r_{g^{-1}}^* \alpha_i \otimes \text{Ad}_{g^{-1}}(X_i). \quad (6.1.7)$$

This defines a connection, known as the *Maurer-Cartan connection*, for the principal  $G$ -bundle  $G \rightarrow \text{pt}$ .

This example can be used to show that any principal  $G$ -bundle  $P$  admits a connection. To that end, we may assume, owing to local triviality, that  $P = M \times G$ . Now take the Maurer-Cartan connection on the principal  $G$ -bundle  $G \rightarrow \text{pt}$ , and take its pullback along the projection  $M \times G \rightarrow G$ ,  $(m, k) \mapsto k$ .

**6.1.8 BASIC FORMS.** Let  $\nu : G \rightarrow \text{Aut}(E)$  be a finite-dimensional representation. An  $E$ -valued differential form  $\eta$  on  $P$  (that is, an element of  $\Omega(P) \otimes E$ ) is said to be *horizontal* if the interior product  $\iota_Y \eta$  vanishes for all vertical vector fields  $Y$  on  $P$ ; it is said to be *invariant* (or *equivariant*) if  $\nu(g^{-1}) \circ \eta = r_g^* \eta$  for all  $g$  in  $G$ . We say that  $\eta$  is *basic* if it is horizontal and invariant.

The significance of basic forms is that they are the forms that can be identified with differential forms on the base space  $M$  that takes values in the associated vector bundle  $P \times_\nu E$ . To be more precise, a differential  $k$ -form  $\omega$  on  $M$  is said to have values in  $P \times_\nu E$  if  $\omega(X_1, \dots, X_k)$  is a section of  $P \times_\nu E$  for any vector fields  $X_1, \dots, X_k$

on  $M$ . We denote by  $\Omega(M; P \times_v E)$  the space of differential forms on  $M$  with values in  $P \times_v E$ . Let  $\Omega_{\text{bas}}(P; E)$  be the space of basic differential forms on  $P$  with values in  $E$ . Then, the pullback along the projection  $P \rightarrow M$  gives a isomorphism of graded spaces

$$\Omega^*(M; P \times_v E) \xrightarrow{\sim} \Omega_{\text{bas}}^*(P; E). \quad (6.1.9)$$

For a proof, see [63, Ex. 5.2, p. 76]. In the special case where  $\nu : G \rightarrow \text{Aut}(\mathbb{R})$  is the trivial 1-dimensional representation, we get

$$\Omega^*(M) \simeq \Omega_{\text{bas}}^*(P). \quad (6.1.10)$$

**6.1.11 COVARIANT DERIVATIVE.** The usual exterior derivative on differential forms do not preserve the space of basic forms. The invariance is not the issue since the exterior derivative is functorial; rather, it is the horizontal property that the exterior derivative can destroy. To remedy this, define the (*exterior*) *covariant derivative* on  $\eta$  in  $\Omega(P; E)$  by

$$D\eta := (d\eta) \circ h,$$

where  $d$  is the usual exterior derivative on  $\Omega(P)$  and  $h : TP \rightarrow HP$  is the projection onto the horizontal subspace. The covariant derivative clearly preserves the subspace  $\Omega_{\text{bas}}(P; E)$  of basic forms.

*Remark.* By the linear isomorphism 6.1.9, the covariant derivative induces a differential  $\nabla$  on the  $(P \times_v E)$ -valued forms on  $M$  so that we would have the following commutative diagram:

$$\begin{array}{ccc} \Omega_{\text{bas}}^*(P; E) & \xrightarrow{D} & \Omega_{\text{bas}}^{*+1}(P; E) \\ \uparrow (6.1.9) \wr & & \wr \uparrow (6.1.9) \\ \Omega^*(M; P \times_v E) & \xrightarrow{\nabla} & \Omega^{*+1}(M; P \times_v E) \end{array}$$

We call  $\nabla$  the (*exterior*) *covariant derivative* on  $P \times_v E$  induced by the connection  $\theta$  on  $P$ . Now the homogenous component  $\Omega^0(M; P \times_v E)$  of  $\Omega^*(M; P \times_v E)$  is simply the space  $\Gamma(P \times_v E)$  of sections of  $P \times_v E$ . The restriction of  $\nabla$  to this subspace yields a linear map,

$$\begin{array}{ccc} \nabla : \Gamma(P \times_v E) & \rightarrow & \Omega^1(M; P \times_v E), \\ \sigma & \mapsto & \nabla \sigma, \end{array} \quad (6.1.12)$$

such that, if we write  $(\nabla \sigma)(X) := \nabla_X \sigma$  for vector fields  $X$  on  $M$ , then

- (i)  $\nabla_{X+\phi Y} \sigma = \nabla_X \sigma + \phi \nabla_Y \sigma$
- (ii)  $\nabla_X (\phi \sigma) = (X\phi) \sigma + \phi \nabla_X \sigma$

for any  $\phi$  in  $C^\infty(M)$ . This defines a connection for the vector bundle  $P \times_v E$  in the conventional sense.

**6.1.13 CURVATURE.** Of particular interest is the covariant derivative of a connection  $\theta$  on  $P$ :

$$\Theta := D\theta.$$



This is called the *curvature* of  $\theta$ . E. Cartan's so-called (*second*) *structural equation* (see [63, Thm. 5.2, p. 77]) states that

$$\Theta = d\theta + \frac{1}{2}[\theta, \theta]_{\mathfrak{g}}. \quad (6.1.14)$$

The bracket in this equation is the commutator for Lie-algebra valued differential forms, which is defined as follows: Suppose  $\omega \in \Omega^k(P) \otimes \mathfrak{g}$  and  $\tau \in \Omega^\ell(P) \otimes \mathfrak{g}$ . Choose a basis  $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$  for  $\mathfrak{g}$  so that we may write  $\omega = \sum_{i=1}^{\dim \mathfrak{g}} \omega_i \otimes X_i$ ,  $\tau = \sum_{i=1}^{\dim \mathfrak{g}} \tau_i \otimes X_i$ . Then,

$$[\omega, \tau]_{\mathfrak{g}} := \sum_{i,j=1}^{\dim \mathfrak{g}} (\omega_i \wedge \tau_j) \otimes [X_i, X_j]_{\mathfrak{g}} \in \Omega^{k+\ell}(P) \otimes \mathfrak{g}.$$

If  $Y$  and  $Z$  are vector fields on  $P$ , then the structural equation 6.1.14 says that

$$\Theta(Y, Z) = d\theta(Y, Z) + [\theta(Y), \theta(Z)]_{\mathfrak{g}},$$

where  $[\cdot, \cdot]_{\mathfrak{g}}$  is now the Lie bracket in  $\mathfrak{g}$ .

The structural equation shows that the curvature form  $\Theta$  is basic. In particular, it is horizontal. So  $\Theta$  is completely determined by its values on the horizontal vector fields. Let  $Y$  and  $Z$  be horizontal vector fields on  $P$ . Since  $\theta$  is vertical,  $\theta(Y) = 0 = \theta(Z)$ . So the structural equation gives us

$$\Theta(Y, Z) = d\theta(Y, Z).$$

A routine calculation yields (see, for instance, [92, Prop. 2.25(f), p. 70])

$$d\theta(Y, Z) = Y\theta(Z) - Z\theta(Y) - \theta([Y, Z]).$$

The first two terms on the right-hand side are zero; thus, we have

$$\Theta(Y, Z) = -\theta([Y, Z]). \quad (6.1.15)$$

**6.1.16 CARTAN'S VIEW ON CONNECTIONS.** It is worth while to mention H. Cartan's [20] point of view on connections. We proceed in three steps:

*Step 1:* Simplify the notation by writing  $\iota_X$  and  $L_X$ , respectively, for the interior product  $\iota_{\tilde{X}}$  and the Lie derivative  $L_{\tilde{X}}$  with respect to the fundamental vector field  $\tilde{X}$ . Equation 6.1.5 can then be written as

$$\iota_X \theta = X. \quad (6.1.17)$$

For Equation 6.1.6, substitute  $g$  with  $\exp(tX)$  and take the derivative with respect to  $t$  at  $t = 0$ ; we get

$$\text{ad}_X \circ \theta = L_X \theta. \quad (6.1.18)$$

*Step 2:* Owing to the canonical isomorphism  $\mathfrak{g} \simeq (\mathfrak{g}^*)^*$ , the  $\mathfrak{g}$ -valued 1-form  $\theta$  can be viewed as a 1-form valued linear function  $\underline{\theta}$  on  $\mathfrak{g}^*$ . Then Equations 6.1.17 and 6.1.18 translates as follows: For

any  $\phi$  in  $\mathfrak{g}^*$ ,

$$\iota_X \underline{\theta}(\phi) = \langle X, \phi \rangle, \quad \underline{\theta} \circ \check{\text{ad}}_X = L_X \underline{\theta}, \quad (6.1.19)$$

where  $\check{\text{ad}} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$  is the dual representation of  $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ , that is,  $\langle \check{\text{ad}}_X \phi, Y \rangle = \langle \phi, \text{ad}_X Y \rangle$  for  $Y$  in  $\mathfrak{g}$ .

*Step 3:* By the universal property of the exterior algebra, the linear map  $\underline{\theta}$  extends to an algebra homomorphism

$$\underline{\theta} : \wedge(\mathfrak{g}^*) \rightarrow \Omega(P). \quad (6.1.20)$$

Now set, for  $X$  in  $\mathfrak{g}$  and  $\phi$  in  $\mathfrak{g}^*$ ,

$$\iota_X \phi := \langle X, \phi \rangle, \quad L_X \phi = \check{\text{ad}}_X \phi.$$

These operations extend to  $\wedge(\mathfrak{g}^*)$  as graded derivations of degree  $-1$  and  $0$ , respectively. Then, owing to equations in 6.1.19,  $\iota_X$  and  $L_X$  commutes with the extended map  $\underline{\theta}$ :

$$\iota_X \circ \underline{\theta} = \theta \circ \iota_X, \quad L_X \circ \underline{\theta} = \underline{\theta} \circ L_X.$$

This is the point of view of H. Cartan on connections — a graded algebra homomorphism  $\wedge(\mathfrak{g}^*) \rightarrow \Omega(P)$  that commutes with the contractions  $\iota_X$  and the Lie derivatives  $L_X$ . In fact, he goes on and defines [20, p. 21] an *algebraic connection* (*connexion algébrique*) as a graded algebra homomorphism from  $\wedge(\mathfrak{g}^*)$  to a  $\mathfrak{g}$ -differential algebra  $\mathcal{E}$  that commutes with  $\iota_X$  and  $L_X$  for all  $X$  in  $\mathfrak{g}$ . The map 6.1.20 is a special case of an algebraic connection.

**6.1.21 CHERN-WEIL THEORY.** Historically, the classical Weil algebra

$$W(\mathfrak{g}) = \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$$

was designed as a universal  $\mathfrak{g}$ -differential algebra that serves as a model space for the connections and their curvatures on principal  $G$ -bundles. More precisely, for any algebraic connection  $\underline{\theta} : \wedge(\mathfrak{g}^*) \rightarrow \mathcal{E}$ , there is a unique  $\mathfrak{g}$ -differential algebra homomorphism

$$CW : W(\mathfrak{g}) \rightarrow \mathcal{E}$$

that makes the following diagram commutative:

$$\begin{array}{ccc} & & W(\mathfrak{g}) \\ & \nearrow \text{inclusion} & \downarrow CW \\ \wedge(\mathfrak{g}^*) & \xrightarrow{\underline{\theta}} & \mathcal{E} \end{array}$$

This was proved by H. Cartan in [20]. Because  $CW$  is a homomorphism of  $\mathfrak{g}$ -differential algebras, it maps basic elements to basic elements; so we have a homomorphism

$$W(\mathfrak{g})_{\text{bas}} \xrightarrow{CW} \mathcal{E}_{\text{bas}}.$$

This, then, induces a homomorphism of cohomologies:

$$H^*\{W(\mathfrak{g})_{\text{bas}}\} \xrightarrow{\text{CW}^*} H^*\{\mathcal{E}_{\text{bas}}\}. \quad (6.1.22)$$

Consider the case where  $\mathcal{E} = \Omega(P)$ ; then, we have

$$H^*\{W(\mathfrak{g})_{\text{bas}}\} \xrightarrow{\text{CW}^*} H^*\{\Omega_{\text{bas}}(P)\}. \quad (6.1.23)$$

By the isomorphism 6.1.10 and de Rham's theorem, we have

$$H^*\{\Omega_{\text{bas}}(P)\} \simeq H^*\{\Omega(M)\} \simeq H^*(M). \quad (6.1.24)$$

Meanwhile, as we have seen in Section 5.3.19,

$$H^*\{W(\mathfrak{g})_{\text{bas}}\} = W(\mathfrak{g})_{\text{bas}} = S(\mathfrak{g}^*)^{\mathfrak{g}}. \quad (6.1.25)$$

So, by Equations 6.1.23–6.1.25, we have an algebra homomorphism

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \rightarrow H^*(M).$$

This is known as the *Chern-Weil homomorphism* [24, 93]. The following is the pinnacle of the Chern-Weil theory on characteristic classes:

**6.1.26 THEOREM.** *Let  $\theta$  be a connection on a principal  $G$ -bundle  $P \rightarrow M$ . Let*

$$\begin{aligned} S(\mathfrak{g}^*)^{\mathfrak{g}} &\rightarrow H^*(M) \\ f &\mapsto [f(P)] \end{aligned} \quad (6.1.27)$$

*be the Chern-Weil homomorphism.*

- (a) *This map is independent of the connection.*
- (b)  *$[f(P)]$  is a characteristic class of  $P$ .*
- (c) *Let  $\Theta$  be the curvature form of  $\theta$ . Let  $f$  in  $S(\mathfrak{g}^*)^{\mathfrak{g}}$  be an invariant homogeneous polynomial on  $\mathfrak{g}$  of degree  $n$ . Then  $f(P)$  is a  $2n$ -form on  $M$  given as follows: For tangent vectors  $X_1, \dots, X_{2n}$  at  $x$  in  $M$ ,*

$$\begin{aligned} f(P)(X_1, X_2, \dots, X_{2n-1}, X_{2n}) = \\ \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) f(\Theta(\bar{X}_{\sigma(1)}, \bar{X}_{\sigma(2)}), \dots, \Theta(\bar{X}_{\sigma(2n-1)}, \bar{X}_{\sigma(2n)})), \end{aligned} \quad (6.1.28)$$

*where  $\bar{X}_i$ ,  $1 \leq i \leq 2n$ , are the lifts of  $X_i$  to the horizontal subspace of  $T_p P$  for some fixed  $p$  in the fiber over  $x$ .*

*Proof.* See [76, Thm. 6.47, p. 278, Lem. 6.49, p. 281]. □

**6.1.29** There is also a Chern-Weil homomorphism for vector bundles. A representation  $\nu : G \rightarrow \text{Aut}(E)$  induces a Lie algebra representation  $\nu_* : \mathfrak{g} \rightarrow \text{End}(E)$  (see Section 2.2.5). Since the curvature  $\Theta$  of a connection on  $P$  is a basic  $\mathfrak{g}$ -valued 2-form on  $P$ , we may apply the

Lie algebra representation  $\nu_*$  to the  $\mathfrak{g}$ -factor of  $\Theta$ ; this<sup>1</sup> gives us a basic  $\text{End}(E)$ -valued 2-form

$$\Theta_E := -\nu_*\Theta$$

on  $P$ . (The negative sign is necessary for  $\Theta_E$  to match with the usual conventions; see the discussion in [35, surrounding Eq. 6.9, p. 56–57].) Now let  $\text{Inv}(E)$  denote the space of polynomials on  $\text{End}(E)$  that are invariant under the conjugation action of  $\text{Aut}(E)$  on  $\text{End}(E)$ . Then, replacing  $\Theta$  with  $\Theta_E$  in the formula 6.1.28 defines the Chern-Weil homomorphism for vector bundles,

$$\begin{aligned} \text{Inv}(E) &\rightarrow H^*(M), \\ f &\mapsto [f(P)]. \end{aligned}$$

Just like the principal bundle case, this map does not depend on the choice of the connection, and it maps each  $f$  in  $\text{Inv}(E)$  to a characteristic class of  $P \times_\nu E$ . We will not dwell on this any further; for details, see [85, Ch.2].

*Remark.* A common practice is to put an extra factor of  $(2\pi i)^{-1}$  in front of each  $\Theta$  in Equation 6.1.28 to make the characteristic class  $[f(P)]$  integral, that is,  $[f(P)] \in H^*(M, \mathbb{Z})$ . Details, in the case of the complex line bundles, can be found in [35, § 15.3].

## 6.2 CLIFFORD MODULE BUNDLES AND SPIN MANIFOLDS

6.2.1 In this section,  $M$  is an  $n$ -dimensional manifold equipped with a Riemannian metric  $\langle \cdot, \cdot \rangle$ .

6.2.2 CLIFFORD BUNDLES. Let  $p$  be a point in  $M$ . Let  $\text{Cl}(T_p M)$  be the Clifford algebra (see Section 5.1.2) generated by  $T_p M$  with the inner product  $\langle \cdot, \cdot \rangle_p$ . It is isomorphic to the Clifford algebra  $\text{Cl}(n)$  generated by  $n$ -dimensional Euclidean space. There is an action of  $O(n)$  on  $\text{Cl}(n)$  that arises from the canonical action of  $O(n)$  on  $\mathbb{R}^n$ . The associated vector bundle

$$\text{Cl}(M) := \text{Fr}_O(M) \times_{O(n)} \text{Cl}(n)$$

is called the *Clifford bundle* of  $M$ . The complex version is defined as

$$\mathbb{C}\text{Cl}(M) := \text{Fr}_O(M) \times_{O(n)} \mathbb{C}\text{Cl}(n).$$

If  $M$  is orientable, we may use the principal  $\text{SO}(n)$ -bundle  $\text{Fr}_{\text{SO}}(M)$  of oriented frames instead of  $\text{Fr}_O(M)$ . Since  $\text{Cl}(M)$  is a vector bundle with fibers isomorphic to  $\text{Cl}(n)$  (see the example on page 93), and since  $\text{Cl}(n)$  is isomorphic to  $\text{Cl}(T_p M)$ , we see that there is a bijection between  $\text{Cl}(M)$  and the disjoint union  $\bigsqcup_{p \in M} \text{Cl}(T_p M)$ .

<sup>1</sup> The corresponding 2-form in  $\Omega(M; P \times_G \text{End}(E))$  is the curvature 2-form for the vector bundle  $P \times_\nu E$  with respect to the induced connection 6.1.12.

6.2.3 CLIFFORD MODULE BUNDLES. A *Clifford module bundle* is a complex vector bundle  $S \rightarrow M$  such that the fiber  $S_p$  over  $p$  in  $M$  is a module over  $\text{Cl}(T_p M)$  in a smooth manner so that we have a  $C^\infty(M)$ -linear operation of smooth sections:  $\Gamma(\text{Cl}(M)) \times \Gamma(S) \rightarrow \Gamma(S)$ ,  $(\psi, \sigma) \mapsto \psi \cdot \sigma$ . Put in another way, there is a bundle morphism

$$c : TM \rightarrow \text{End}(S) \simeq S^* \otimes S$$

as real vector bundles, such that for any vector field  $X$  on  $M$ ,

$$c(X) \circ c(X) = \frac{1}{2} \langle X, X \rangle \mathbf{1}$$

where  $\mathbf{1}$  is the identity map on  $S$ . The map  $c$  then extends to  $c : \text{Cl}(M) \rightarrow \text{End}(S)$ , which is called the *Clifford action* of  $\text{Cl}(M)$  on  $S$ .

A *Clifford connection* on the Clifford module bundle  $S$  is a connection  $\nabla$  on  $S$  that is compatible with the Riemannian<sup>2</sup> connection on  $M$ , which we also denote by  $\nabla$ , in the sense that

$$[\nabla_X, c(Y)] = c(\nabla_X Y)$$

for any vector fields  $X$  and  $Y$  on  $M$ . If the Clifford bundle  $S$  is equipped with a Hermitian metric  $(\cdot, \cdot)$ , we further require that the Clifford action and the Clifford connection satisfy the following properties.

- (i) The Clifford action of tangent vectors leaves the metric invariant:

$$(c(X)\sigma_1, c(X)\sigma_2) = (\sigma_1, \sigma_2).$$

- (ii) The Clifford connection is a metric connection:

$$X(\sigma_1, \sigma_2) = (\nabla_X \sigma_1, \sigma_2) + (\sigma_1, \nabla_X \sigma_2).$$

Recall that the image of  $\wedge^2(T_p M)$  under the quantization map  $q : \wedge T_p M \rightarrow \text{Cl}(T_p M)$  is the Lie algebra  $\mathfrak{spin}(T_p M)$  in  $\text{Cl}(T_p M)$  (see Section 5.1.40). Property (i) guarantees that the  $\text{Spin}(T_p M)$ -action on  $S$  induced by the Clifford action is unitary. A pair  $(\langle \cdot, \cdot \rangle, \nabla)$  of a Hermitian metric and a compatible Clifford connection for a Clifford module bundle  $S$  is called a *Dirac structure* for  $S$ . Any Clifford module bundle admits a Dirac structure; if  $S$  is trivial so that  $S = M \times E$ , then the existence of a Dirac structure follows from the fact that (a) the  $\text{Cl}(n)$ -module  $E$  admits an inner product that is invariant under the action of the generators in  $\mathbb{R}^n$  (see Section 5.1.44) and (b) any vector bundle with a Hermitian metric admits a metric connection (see [63, p. 118]); the existence of a Dirac structure for an arbitrary Clifford module bundle follows, owing to the existence of the partition of unity subordinate to the locally trivializing open cover for  $M$ .

<sup>2</sup> Also known as the Levi-Civita connection. It satisfies  $\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$  (it is a metric connection) and  $\nabla_X Y - \nabla_Y X = [X, Y]$  (torsion free). The fundamental theorem of Riemannian Geometry states that there is a unique Riemannian connection on a Riemannian manifold; see [22, p. 1–2].

What will occupy most of our discussions is the case where the Clifford module bundle splits as a vector bundle into

$$S = S^+ \oplus S^-,$$

and the Clifford action by a vector field  $X$  on  $M$  is an *odd* endomorphism that maps  $\Gamma(S^+)$  to  $\Gamma(S^-)$  and vice versa. Such Clifford module bundles are said to be  $\mathbb{Z}/2\mathbb{Z}$ -graded. If that is the case, then we demand that the Clifford connection respects the decomposition.

**6.2.4 SPIN MANIFOLDS.** A *spin manifold* is an oriented Riemannian manifold with an extra structure that provides a standard procedure to construct Clifford module bundles; that structure — called the *spin structure* — is the existence of a principal  $\text{Spin}(n)$ -bundle  $P_{\text{Spin}}(M)$  and a bundle map

$$\alpha : P_{\text{Spin}}(M) \rightarrow \text{Fr}_{\text{SO}}(M)$$

that is  $\text{Spin}(n)$ -equivariant in the sense that, for  $x$  in  $P_{\text{Spin}}(M)$  and  $g$  in  $\text{Spin}(n)$ ,

$$\alpha(g \cdot x) = \pi(g) \cdot \alpha(x)$$

where  $\pi$  is the canonical double covering  $\text{Spin}(n) \rightarrow \text{SO}(n)$ . Note that  $\alpha$  is a double covering when restricted to a fiber in  $P_{\text{Spin}}(M)$ . A manifold is orientable if and only if its 1st Stiefel-Whitney class in  $H^1(M; \mathbb{Z}/2\mathbb{Z})$  is zero; it admits a spin structure if and only if its 2nd Stiefel-Whitney class in  $H^2(M; \mathbb{Z}/2\mathbb{Z})$  is zero; see [69, Thm. 1.2, p. 79, Thm. 1.7, p. 82].

Suppose  $M$  is a spin manifold. Then we may redefine the Clifford bundle of  $M$  as

$$\text{Cl}(M) = P_{\text{Spin}}(M) \times_{\text{Spin}(n)} \text{Cl}(n).$$

The identification of the fiber  $\text{Cl}(M)_p$  over  $p$  in  $M$  with  $\text{Cl}(T_p M)$  is done similarly as in Section 6.2.2. Now suppose  $E$  is a finite-dimensional  $\text{Cl}(n)$ -module. Then the associated vector bundle

$$E(M) := P_{\text{Spin}}(M) \times_{\text{Spin}(n)} E$$

is a Clifford module bundle. To see this, take an element of the fiber  $E(M)_p$  over  $p$ , which is an equivalence class  $[f, v]$  of  $(f, v)$  in  $P_{\text{Spin}}(M) \times E$ . An element of the fiber  $\text{Cl}(M)_p$ , likewise, is an equivalence class  $[f', w]$ . One can find an element  $g$  in  $\text{Spin}(n)$  such that  $f' \cdot g = f$ . Then,  $[f', w] = [f, g^{-1}w]$  in  $\text{Cl}(M)_p$ ; its Clifford action on  $[f, v]$  in  $E(M)_p$  is  $c([f, g^{-1}w])[f, v] = [f, c(g^{-1}w)v]$ .

There is a Clifford connection for  $E(M)$  induced by the Riemannian connection on  $M$ ; here is how it goes: The tangent bundle  $TM$  is isomorphic to the associated vector bundle  $\text{Fr}_{\text{SO}}(M) \times_{\text{SO}(n)} \mathbb{R}^n$ . So there is a connection on the principal bundle  $\text{Fr}_{\text{SO}}(M)$  that corresponds to the Riemannian connection on  $TM$ . That connection on  $\text{Fr}_{\text{SO}}(M)$  then induces a connection on the Clifford module bundle  $P_{\text{Spin}}(M) \times_{\text{Spin}(n)} E$ . This is a Clifford connection on  $E(M)$ ; see

[69, Prop. 4.11, p. 108].

In the special case where  $E$  is the space  $\mathbb{S}$  of  $n$ -spinors, we have the vector bundle

$$\mathbb{S}(M) := P_{\text{Spin}}(M) \times_{\text{Spin}(n)} \mathbb{S}.$$

This is called the *spinor bundle* of  $M$ .

6.2.5 If  $M$  is even-dimensional, any Clifford module bundle is (owing to Equation 5.1.7) of the form

$$\mathbb{S}(M) \otimes W$$

where  $W$  is some auxiliary complex vector bundle on which the Clifford bundle  $\mathbb{C}l(M)$  acts trivially. Also, since the spinor space has a natural  $\mathbb{Z}/2\mathbb{Z}$ -grading (see Section 5.1.44), the Clifford module bundle is  $\mathbb{Z}/2\mathbb{Z}$ -graded.

6.2.6 DIRAC OPERATORS. Let  $M$  be a spin manifold and  $\nabla$  be the Clifford connection on the Clifford module bundle  $E(M)$ . The *geometric Dirac operator* is the 1st order differential operator on the vector bundle  $E(M)$  which is locally expressed as

$$\mathcal{D} = \sum_{i=1}^n c(e_i) \nabla_{e_i}, \quad (6.2.7)$$

where  $(e_1, \dots, e_n)$  is a local orthonormal framing of the tangent bundle of  $M$ . If the connection  $\nabla$  is induced by the Riemannian connection on  $M$ , then the geometric Dirac operator is said to be *Riemannian*. The geometric Dirac operator is globally defined and independent of the choice for the framing because it is equal to the following composition of operations:

$$\mathcal{D} : \Gamma(E(M)) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\langle \cdot, \cdot \rangle} \Gamma(TM \otimes E) \xrightarrow{c} \Gamma(E(M)).$$

If the bundle  $E(M)$  is  $\mathbb{Z}/2\mathbb{Z}$ -graded, its space of sections is naturally graded:

$$\Gamma(E(M)) = \Gamma(E(M)^+) \oplus \Gamma(E(M)^-).$$

(This happens when  $M$  is even-dimensional, as pointed out in Section 6.2.5.) In this case, the Dirac operator is an odd operator because of our requirement that the Clifford action of a tangent vector be odd (see Section 6.2.3).

Geometric Dirac operators belong to the more general class of *Dirac operators* [12, Def. 3.36, p. 116]; a Dirac operator  $\mathcal{D}$  on  $M$  is a 1st-order differential operator on a vector bundle over  $M$  (an odd operator if the bundle is  $\mathbb{Z}/2\mathbb{Z}$ -graded) such that its square is a *generalized Laplacian*, which means that  $\mathcal{D}^2$  is a 2nd-order differential operator whose expression in local coordinates is of the form

$$\mathcal{D}^2 = \frac{1}{2} \sum_{i,j} \eta^{ij} \partial_i \partial_j + (\text{lower order part}),$$

where  $\eta^{ij}$  is the  $(i, j)$ -entry of the inverse of the matrix  $[\eta_{ij}]$  coming from the metric  $\eta = \sum_{i,j} \eta_{ij} dx^i dx^j$  on  $M$ .

**6.2.8 SPIN STRUCTURE ON HOMOGENEOUS SPACES.** We now focus our attention to the case  $M = G/K$  where  $G$  is a Lie group (not necessarily compact) and  $K$  is a closed connected subgroup. Assume that  $G$  admits a bi-invariant metric, that is, there is an  $\text{Ad}(G)$ -invariant inner product on  $\mathfrak{g}$ . Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be an orthogonal decomposition. Let  $k$  be an arbitrary element of  $K$ . Clearly,  $\text{Ad}_k(\mathfrak{k}) \subseteq \mathfrak{k}$ . Then  $\text{Ad}_k(\mathfrak{p}) \subseteq \mathfrak{p}$  because

$$0 = \langle \text{Ad}_k^{-1}(\mathfrak{k}), \mathfrak{p} \rangle = \langle \mathfrak{k}, \text{Ad}_k(\mathfrak{p}) \rangle.$$

Thus,  $\mathfrak{p}$  is a  $K$ -vector space, and since  $K$  is connected, this representation maps  $K$  into  $\text{SO}(\mathfrak{p})$ .

We view  $G$  as a principal  $K$ -bundle over  $G/K$ . The vector bundle associated to the representation

$$K \xrightarrow{\text{Ad}} \text{SO}(\mathfrak{p}) \quad (6.2.9)$$

is (isomorphic to) the tangent bundle of  $G/K$ :

$$T(G/K) \simeq G \times_{\text{Ad}} \mathfrak{p}.$$

This gives us means to identify the tangent space at any point in  $G/K$  with  $\mathfrak{p}$ . The set of orthonormal frames for  $\mathfrak{p}$  can be identified with  $\text{SO}(\mathfrak{p})$ . Let  $K$  act on  $\text{SO}(\mathfrak{p})$  on the left via the representation 6.2.9. The associated vector bundle is (isomorphic to) the orthonormal oriented frame bundle:

$$\text{Fr}_{\text{SO}}(G/K) \simeq G \times_K \text{SO}(\mathfrak{p}).$$

Seeking a spin structure for  $G/K$ , suppose the orthogonal action of  $K$  on  $\mathfrak{p}$  can be lifted to  $\text{Spin}(\mathfrak{p})$ :

$$\begin{array}{ccc} & \text{Spin}(\mathfrak{p}) & \\ \nearrow \widetilde{\text{Ad}} & \downarrow & \\ K & \xrightarrow{\text{Ad}} & \text{SO}(\mathfrak{p}) \end{array} \quad (6.2.10)$$

Let  $K$  act on  $\text{Spin}(\mathfrak{p})$  on the left via  $\widetilde{\text{Ad}}$ . Then the mixing construction

$$P_{\text{Spin}}(G/K) := G \times_K \text{Spin}(\mathfrak{p})$$

gives a principal  $\text{Spin}(\mathfrak{p})$ -bundle over  $G/K$ , and there is an obvious bundle map

$$P_{\text{Spin}}(G/K) \rightarrow \text{Fr}_{\text{SO}}(G/K).$$

So we have a spin structure on  $G/K$ . Conversely, a spin structure on  $G/K$  implies the existence of the lift 6.2.10 if  $G$  is simply connected [11, Lem. 3, p. 66] or  $G$  is compact connected with  $\dim(G/K) \geq 3$  [68, p. 14].



### 6.3 EQUIVARIANT DIFFERENTIAL OPERATORS AND THE RELATIVE WEIL ALGEBRA

6.3.1 We continue with the assumptions made in Section 6.2.8. Let  $\nu : K \rightarrow \text{Aut}(E)$  be a finite-dimensional representation. By the linear isomorphism 6.1.9, we have an isomorphism of graded spaces,

$$\Omega_{\text{bas}}^*(G; E) \xrightarrow{\sim} \Omega^*(G/K; G \times_{\nu} E). \quad (6.3.2)$$

At the level of degree 0, we have on the left-hand side the space of  $E$ -valued functions that are  $K$ -invariant, and on the right-hand side the space of sections of  $G \times_{\nu} E$ . So we have a linear isomorphism

$$\Xi : \Gamma(G \times_{\nu} E) \xrightarrow{\sim} (C^{\infty}(G) \otimes E)^K. \quad (6.3.3)$$

The action of  $K$  on  $C^{\infty}(G) \otimes E$  is by  $R \otimes \nu$  where  $R$  denotes the right-regular action. Let us spell out how the isomorphism  $\Xi$  works. Let  $\bar{\sigma}$  be a section of  $G \times_{\nu} E$ . Its value at  $gK$  is an equivalence class  $[g, w_g]$  for some  $w_g$  in  $E$ . Had we chosen a different representative for  $[g, w_g]$ , say  $(gk, w_{gk})$ , then since  $[gk, w_{gk}] = [g, \nu(k)w_{gk}]$  for any  $k$  in  $K$ , we have  $[g, w_g] = [g, \nu(k)w_{gk}]$ , which implies

$$\nu(k)w_{gk} = w_g. \quad (6.3.4)$$

Now define the  $E$ -valued function  $\sigma$  on  $G$  by

$$\sigma(g) = w_g.$$

Then,

$$\begin{aligned} \sigma(gk) &= w_{gk} \\ &= \nu(k)^{-1}w_g && \text{(by Equation 6.3.4)} \\ &= \nu(k)^{-1}\sigma(g). \end{aligned}$$

Hence,  $\sigma$  is an element of  $(C^{\infty}(G) \otimes E)^K$ ; this is exactly the image of  $\bar{\sigma}$  under  $\Xi$ .

6.3.5 We wish to identify an element of the relative Weil algebra (see Section 5.4 for definitions and notations)

$$\mathcal{W}(\mathfrak{g}, \mathfrak{k}) := (\mathcal{U}(\hat{\mathfrak{g}}) \otimes \text{Cl}(\mathfrak{p}))^{\mathfrak{k}}$$

as a differential operator on  $(C^{\infty}(G) \otimes E)^K$ . The most natural way to proceed is to use the algebra isomorphism

$$\tau \otimes \mathbf{1} : \mathcal{U}(\hat{\mathfrak{g}}) \otimes \text{Cl}(\mathfrak{p}) \xrightarrow{\sim} D(G) \otimes \text{Cl}(\mathfrak{p}) \quad (6.3.6)$$

where  $\tau$  is the algebra isomorphism 2.2.11 (here we use the fact that  $\hat{\mathfrak{g}}$  is a copy of  $\mathfrak{g}$ ) and  $\mathbf{1}$  is the identity automorphism of  $\text{Cl}(\mathfrak{p})$ . This means that an element of  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  acts on  $(C^{\infty}(G) \otimes E)^K$  as follows:

- (i) The action of  $\text{Cl}(\mathfrak{p})$ -factor on  $(C^{\infty}(G) \otimes E)^K$  is by the Clifford module structure on  $E$ .
- (ii) For the action of  $\mathcal{U}(\hat{\mathfrak{g}})$ -factor, it is sufficient to know the ac-

tion of the generators in  $\hat{\mathfrak{g}}$ ; for  $X$  in  $\mathfrak{g}$ , the generator  $\hat{X}$  acts as the directional derivative  $\partial_X$  with respect to the left-invariant vector field on  $G$  generated by  $X$ . In other words, for  $\sigma \in (C^\infty(G) \otimes E)^K$  and  $g \in G$ ,

$$(\hat{X}\sigma)(g) = \left. \frac{d}{dt} \right|_0 \sigma(g \exp(tX)). \quad (6.3.7)$$

Let  $G$  act on  $(C^\infty(G) \otimes E)^K$  by  $L \otimes \mathbf{1}$  where  $L$  is the left-regular representation on  $C^\infty(G)$  and  $\mathbf{1}$  is the trivial representation on  $E$ . Then, the differential operators on  $(C^\infty(G) \otimes E)^K$  that are obtained from  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  in the above manner are  $G$ -equivariant.

*Remark.* Note that there is no mention of connections in the above identification procedure. Though it is possible to introduce a suitable<sup>3</sup> connection on the trivial bundle  $G \times E \rightarrow G$  for this task, we prefer not to, keeping the use of differential geometry to the minimum.

6.3.8 Let  $Y$  be a vector in  $\mathfrak{p}$ . Then,  $\hat{Y}$  in  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  acts on  $(C^\infty(G) \otimes E)^K$  as a differential operator according to Equation 6.3.7. We wish to define the action of  $\hat{Y}$  on  $\Gamma(G \times_{\mathfrak{v}} E)$  so that the following diagram commutes:

$$\begin{array}{ccc} (C^\infty(G) \otimes E)^K & \xrightarrow{\hat{Y}} & (C^\infty(G) \otimes E)^K \\ \Xi \uparrow \wr & & \wr \uparrow \Xi \\ \Gamma(G \times_{\mathfrak{v}} E) & \xrightarrow{\hat{Y}} & \Gamma(G \times_{\mathfrak{v}} E) \end{array} \quad (6.3.9)$$

We need to specify the value  $\hat{Y}\bar{\sigma}$  at  $gK$ , where  $g$  is an arbitrary element of  $G$ . To that end, let  $t \mapsto w(t)$  be a curve in  $E$  defined over some interval containing 0, such that

$$\bar{\sigma}(g \exp(tY)K) = [g \exp(tY), w(t)] \in G \times_{\mathfrak{v}} E. \quad (6.3.10)$$

This is possible because

$$\begin{array}{l} \mathfrak{p} \rightarrow G/K, \\ X \mapsto g \exp(X)K, \end{array} \quad (6.3.11)$$

is a local diffeomorphism near 0 (see [54, Lem. 4.1, p. 123]). Then, we define the value of  $\hat{Y}\bar{\sigma}$  at  $gK$  to be

$$(\hat{Y}\bar{\sigma})(gK) := [g, w'(0)]. \quad (6.3.12)$$

We claim that this is the action of  $\hat{Y}$  on  $\Gamma(G \times_K E)$  that makes the diagram 6.3.9 commutative. Indeed, let  $\sigma := \Xi(\bar{\sigma})$ . Then  $\sigma$  satisfies

<sup>3</sup> We discussed this matter briefly on pages 81 and 82. To recapitulate, if  $\mathfrak{k} = 0$ , then take the  $\frac{1}{3}\gamma$  term that appears in Equation 5.2.22 as the connection 1-form for the vector bundle; if  $\mathfrak{k} \neq 0$ , then use  $\frac{1}{3}\gamma^{\mathfrak{p}}$ . For more discussions in this direction, see [1].

(see Section 6.3.1)

$$\sigma(g \exp(tY)) = w(t);$$

hence,

$$\widehat{Y}\sigma(g) = w'(0).$$

This implies that the value of  $\Xi^{-1}(\widehat{Y}\sigma)$  at  $gK$  is

$$\Xi^{-1}(\widehat{Y}\sigma)(gK) = [g, w'(0)].$$

By Equation 6.3.12, this is equal to  $(\widehat{Y}\bar{\sigma})(gK)$ .

**6.3.13 LEMMA.** *Identify  $\mathfrak{p}$  with the tangent space of  $G/K$  at  $gK$  by the differential of the map 6.3.11 at 0. For  $Y$  in  $\mathfrak{p}$  and  $\bar{\sigma}$  in  $\Gamma(G \times_{\mathbb{V}} E)$ , define  $\bar{\sigma} \mapsto \widehat{Y}\bar{\sigma}$  by Equation 6.3.12. Then  $\widehat{Y}$  is the directional derivative at  $gK$  with respect to the tangent vector  $Y$ .*

*Proof.* Let  $V$  be a neighborhood of 0 in  $\mathfrak{p}$ , and  $U$  a neighborhood of  $gK$  in  $G/K$ , such that  $V$  is mapped diffeomorphically onto  $U$  by the map 6.3.11; denote this diffeomorphism between  $V$  and  $U$  by

$$\psi : V \rightarrow U.$$

The inverse of  $\psi$  yields a coordinate chart near  $gK$ .

Denote the vector bundle projection  $P \times_{\mathbb{V}} E \rightarrow G/K$  by  $\xi$ . Note that the construction of the curve  $w$  by Equation 6.3.10 amounts to a local trivialization

$$\varphi : \xi^{-1}(U) \rightarrow U \times E$$

over  $U$ .

Let  $\bar{\sigma}_{\varphi, \psi}$  be the expression<sup>4</sup> of the section  $\bar{\sigma}$  in the chart  $\psi$  and the trivialization  $\varphi$ . Then, Equation 6.3.10 translates into

$$\bar{\sigma}_{\varphi, \psi}(tY) = w(t).$$

The derivative with respect to  $t$  at  $t = 0$  is precisely the directional derivative of  $\bar{\sigma}_{\varphi, \psi}$  with respect to the tangent vector  $Y$  at 0 in  $\mathfrak{p}$ :

$$(\partial_Y \bar{\sigma}_{\varphi, \psi})(0) = w'(0).$$

Meanwhile, Equation 6.3.12 translates into,

$$(\widehat{Y}\bar{\sigma}_{\varphi, \psi})(0) = w'(0).$$

Therefore,  $\widehat{Y} = \partial_Y$ . □

**6.3.14 PROPOSITION.** *Let  $\widehat{Y} \in \hat{\mathfrak{p}}$  act on  $\bar{\sigma} \in \Gamma(G \times_K E)$  as Equation 6.3.12 so that we have the commutative diagram 6.3.9. Then the action of the relative Dirac operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  on  $(C^\infty(G) \otimes E)^K$  agrees*

<sup>4</sup> Explicitly,  $\bar{\sigma}_{\varphi, \psi} : V \rightarrow E$  is given by the series of compositions,  $V \xrightarrow{\psi} U \xrightarrow{\bar{\sigma}} \xi^{-1}(U) \xrightarrow{\varphi} U \times E \xrightarrow{p_2} E$  where  $p_2$  is the projection onto the second component.

with the action of the following differential operator on  $\Gamma(G \times_v E)$ :

$$\mathcal{D}_{\mathfrak{g}/\mathfrak{k}} := \sum_{i=1}^{\dim \mathfrak{p}} \widehat{Y}_i \otimes Y_i + 1 \otimes \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{p}} Y_i \gamma^{\mathfrak{p}}(Y_i) \in \widehat{\mathfrak{p}} \otimes \text{Cl}(\mathfrak{p}), \quad (6.3.15)$$

where  $\{Y_i\}_{i=1}^{\dim \mathfrak{p}}$  is any orthonormal basis for  $\mathfrak{p}$ .

*Remark.* As the proof demonstrates,  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  is just another expression for the relative Dirac operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ . The element  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  was first introduced by B. Kostant [65]. That it is equal to  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  was observed by A. Alekseev and E. Meinrenken [3].

*Proof.* Owing to our discussion in Section 6.3.8, we may take the right-hand side of Equation 6.3.15 as an element of  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  and show its equality with  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ . In other words, we wish to show that

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k}) = \sum_{i=1}^{\dim \mathfrak{p}} \widehat{Y}_i Y_i + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{p}} Y_i \gamma^{\mathfrak{p}}(Y_i) \quad (6.3.16)$$

in  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$ .

Let  $\{X_i\}_{i=1}^{\dim \mathfrak{k}}$  and  $\{Y_j\}_{j=1}^{\dim \mathfrak{p}}$  be orthonormal basis for  $\mathfrak{k}$  and  $\mathfrak{p}$ , respectively. By Equation 5.2.22, we have

$$\mathcal{D}_{\mathfrak{g}} = \sum_{i=1}^{\dim \mathfrak{k}} \widehat{X}_i X_i + \sum_{j=1}^{\dim \mathfrak{p}} \widehat{Y}_j Y_j + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{g}}(X_i) + \frac{1}{3} \sum_{j=1}^{\dim \mathfrak{p}} Y_j \gamma^{\mathfrak{g}}(Y_j). \quad (6.3.17)$$

By Equation 5.4.12,  $\gamma^{\mathfrak{g}}(X_i) = \gamma^{\mathfrak{k}}(X_i) + \gamma^{\mathfrak{p}}(X_i)$ . So Equation 6.3.17 can be rewritten as

$$\begin{aligned} \mathcal{D}_{\mathfrak{g}} &= \sum_{i=1}^{\dim \mathfrak{k}} \widehat{X}_i X_i + \sum_{j=1}^{\dim \mathfrak{p}} \widehat{Y}_j Y_j \\ &\quad + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{k}}(X_i) + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{p}}(X_i) + \frac{1}{3} \sum_{j=1}^{\dim \mathfrak{p}} Y_j \gamma^{\mathfrak{g}}(Y_j). \end{aligned} \quad (6.3.18)$$

For the factor  $\gamma^{\mathfrak{g}}(Y_j)$  in the last summation above, we have, by Equation 5.4.8,

$$\begin{aligned} \gamma^{\mathfrak{g}}(Y_j) &= - \sum_{i=1}^{\dim \mathfrak{k}} \sum_{k>i}^{\dim \mathfrak{k}} \underbrace{\langle Y_j, [X_i, X_k]_{\mathfrak{g}} \rangle}_{=0} X_i X_k \\ &\quad - \sum_{i=1}^{\dim \mathfrak{k}} \sum_{k=1}^{\dim \mathfrak{p}} \langle Y_j, [X_i, Y_k]_{\mathfrak{g}} \rangle X_i Y_k - \underbrace{\sum_{i=1}^{\dim \mathfrak{p}} \sum_{k>i}^{\dim \mathfrak{p}} \langle Y_j, [Y_i, Y_k]_{\mathfrak{g}} \rangle Y_i Y_k}_{= \gamma^{\mathfrak{p}}(Y_j)} \\ &= \sum_{i=1}^{\dim \mathfrak{k}} \sum_{k=1}^{\dim \mathfrak{p}} \langle X_i, [Y_j, Y_k]_{\mathfrak{g}} \rangle X_i Y_k + \gamma^{\mathfrak{p}}(Y_j). \end{aligned}$$

Thus, for the last summation in Equation 6.3.18, we have

$$\begin{aligned}
& \frac{1}{3} \sum_{j=1}^{\dim \mathfrak{p}} Y_j \gamma^{\mathfrak{g}}(Y_j) \\
&= -\frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} \sum_{j=1}^{\dim \mathfrak{p}} \sum_{k=1}^{\dim \mathfrak{p}} \langle X_i, [Y_j, Y_k]_{\mathfrak{g}} \rangle X_i Y_j Y_k + \frac{1}{3} \sum_{j=1}^{\dim \mathfrak{p}} Y_j \gamma^{\mathfrak{p}}(Y_j) \\
&= \frac{2}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{p}}(X_i) + \frac{1}{3} \sum_{j=1}^{\dim \mathfrak{p}} Y_j \gamma^{\mathfrak{p}}(Y_j).
\end{aligned}$$

Inserting this into Equation 6.3.18, we get

$$\begin{aligned}
\mathcal{D}_{\mathfrak{g}} &= \sum_{i=1}^{\dim \mathfrak{k}} \widehat{X}_i X_i + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{k}}(X_i) \\
&\quad + \sum_{j=1}^{\dim \mathfrak{p}} \widehat{Y}_j Y_j + \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{g}}(X_i) + \frac{1}{3} \sum_{j=1}^{\dim \mathfrak{p}} Y_j \gamma^{\mathfrak{p}}(Y_j). \quad (6.3.19)
\end{aligned}$$

For  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ , we must subtract from the above the image  $\mathcal{D}_{\mathfrak{k}}$  of the cubic Dirac operator  $\sum_{i=1}^{\dim \mathfrak{k}} \widehat{X}_i X_i + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{k}}(X_i)$  in  $\mathcal{W}(\mathfrak{k})$  under the canonical inclusion  $\mathcal{W}\mathfrak{k} \hookrightarrow \mathcal{W}\mathfrak{g}$ . By Lemma 5.4.11(b),

$$\mathcal{D}_{\mathfrak{k}} = \sum_{i=1}^{\dim \mathfrak{k}} \widehat{X}_i X_i + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{k}}(X_i) + \sum_{i=1}^{\dim \mathfrak{k}} X_i \gamma^{\mathfrak{p}}(X_i). \quad (6.3.20)$$

Subtracting this from Equation 6.3.19 gives us Equation 6.3.16.  $\square$

# 7

## THE ASYMPTOTIC EXPANSION OF THE HEAT KERNEL ASSOCIATED TO THE CUBIC DIRAC OPERATOR

LET  $G$  be a compact connected Lie group with a bi-invariant metric, and let  $K$  be a closed connected subgroup. The metric defines an invariant inner product on the Lie algebra  $\mathfrak{g}$  of  $G$ , and the Lie algebra  $\mathfrak{k}$  of  $K$  is a Lie subalgebra of  $\mathfrak{g}$ . The pair  $(\mathfrak{g}, \mathfrak{k})$  defines the relative Dirac operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ ; its square is given by (owing to Equations 5.4.18 and 5.4.19)

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2 = \frac{1}{2} \widehat{\Omega}_{\mathfrak{g}} - \frac{1}{2} \text{diag}_W \widehat{\Omega}_{\mathfrak{k}} - \frac{1}{2} \|\rho_{\mathfrak{g}}\|^2 + \frac{1}{2} \|\rho_{\mathfrak{k}}\|^2.$$

See Notation 5.4.15 and Section 5.4.17 for the notations. This is an element of the relative Weil algebra  $\mathcal{W}(\mathfrak{g}, \mathfrak{k}) := (\mathcal{U}(\widehat{\mathfrak{g}}) \otimes \text{Cl}(\mathfrak{p}))^{\mathfrak{k}}$ . As such, it can be identified as a differential operator via the algebra isomorphism

$$\tau \otimes 1 : \mathcal{U}(\widehat{\mathfrak{g}}) \otimes \text{Cl}(\mathfrak{p}) \xrightarrow{\sim} D(G) \otimes \text{Cl}(\mathfrak{p})$$

defined in Section 6.3.5. The image of  $\widehat{\Omega}_{\mathfrak{g}}$  under  $\tau \otimes 1$  is the Laplacian on  $G$  (this is essentially Equation 2.2.18). But, because of the term  $-\frac{1}{2} \text{diag}_W \widehat{\Omega}_{\mathfrak{k}}$ , the image of  $\mathcal{D}^2(\mathfrak{g}, \mathfrak{k})$  under  $\tau \otimes 1$  is, a priori, not of Laplace type. Still, we shall see that, as an operator on the domain  $(C^\infty(G) \otimes E)^K$  (where  $E$  is a finite-dimensional  $\text{Cl}(\mathfrak{p})$ -module),  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$  is equal to a generalized Laplacian. So it makes sense to consider the heat kernel of  $(\tau \otimes 1)(\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2)$ . Its asymptotic expansion, then, follows essentially from that of the scalar Laplacian on  $G$  (which we calculated in Theorem 4.2.37).

To avoid the cluttering of notations, we shall suppress  $\tau \otimes 1$  in

our discussions; it will be clear from the context whether we are considering an element of  $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$  as is, or as its image under  $\tau \otimes 1$ . Similarly, we shall identify elements in  $W(\mathfrak{g}, \mathfrak{k}) = (S(\hat{\mathfrak{g}}) \otimes \wedge(\mathfrak{p}))^{\mathfrak{k}}$  with elements in  $D(\mathfrak{g}) \otimes \wedge(\mathfrak{p})$ .

## 7.1 THE CONVOLUTION KERNEL OF A GENERALIZED LAPLACIAN

As a Dirac operator  $\mathcal{D}$  is a “square root” of a generalized Laplacian, by the *heat kernel associated to  $\mathcal{D}$*  we mean the heat kernel of the generalized Laplacian  $\mathcal{D}^2$ . In this section, we briefly review some basic facts regarding generalized Laplacians and their heat kernels. It will be a slight generalization of what we saw in Section 2.1.9.

Let  $F$  be a vector bundle over a compact oriented Riemannian manifold  $M$ . We assume that  $F$  is equipped with a fiber-wise inner product in a smooth manner. A *generalized Laplacian* on  $\Gamma(F)$  is a differential operator  $L$  its expression in local coordinates takes the form

$$L = \frac{1}{2} \sum_{i,j=1}^{\dim M} \eta^{ij} \partial_i \partial_j + (\text{lower order part}), \quad (7.1.1)$$

where the matrix  $\eta^{ij}$  is the  $(i, j)$ -entry of the inverse of the matrix  $[\eta_{ij}]$  coming from the metric  $\eta = \sum_{i,j} \eta_{ij} dx^i dx^j$  on  $M$ . The one-half factor above is an insignificant part in our discussion; we put it here on account of Equation 5.4.18.

The heat kernel  $Q_t$  of  $L$  is the integral kernel of the operator  $e^{tL}$  ( $t > 0$ ), that is,

$$(e^{tL}\sigma)(x) = \int_M Q_t(x, y) \sigma(y) \text{vol}_y$$

for any section  $\sigma$  of  $F$ . Here  $\text{vol}_y$  is the Riemannian volume form at  $y$  in  $M$ . Thus,  $Q_t$  is a smooth section of the bundle  $F \boxtimes F^* := p_1^* F \otimes p_2^* F^*$ , where  $p_1$  and  $p_2$ , respectively, are the projections  $M \times M \rightarrow M$  onto the first and second components. For existence of the smooth kernel  $Q_t$ , see [85, Prop. 5.31, p. 83]. The heat kernel  $Q_t$  has the following fundamental properties:

- (i) It satisfies the heat equation:

$$(\partial_t - L)Q_t(x, y) = 0,$$

where  $L$  applies to the variable  $x$ .

- (ii) It converges to the Dirac delta distribution  $\delta(x - y)$  as  $t \rightarrow 0$ , in the sense that, for any smooth section  $\sigma$  of  $F$ ,

$$\int_M Q_t(x, y) \sigma(y) \text{vol}_y \rightarrow \sigma(x)$$

uniformly.

The smooth kernel  $Q_t$  is uniquely determined by the above two properties; see [85, Prop. 7.5, p. 96].

Suppose a compact Lie group  $G$  acts transitively on the manifold  $M$  as isometries. Then we have the identification

$$M \simeq G/K$$

where  $K$  is the stabilizer of a point  $x_0$  in  $M$  of our choice. Let us write

$$\tilde{x} := xK \in G/K.$$

Assume that the bundle  $F$  is homogeneous over  $G/K$ , in the sense that

$$F = G \times_K V$$

for some finite-dimensional  $K$ -vector space  $V$ . There is a natural  $G$ -action on  $F$  given by

$$g' \cdot [g, v] := [g'g, v]$$

for  $g'$  in  $G$  and  $[g, v]$  in  $G \times_K V$ . Denote this action by  $\alpha : G \rightarrow \text{Aut}(F)$ . This induces a  $G$ -action on the space  $\Gamma(F)$  of smooth sections of  $F$ :

$$(g \cdot \sigma)(\tilde{x}) := \alpha_g(\sigma(g^{-1}\tilde{x})).$$

Here  $g^{-1}\tilde{x}$  denotes  $g^{-1}xK$ .

Suppose the generalized Laplacian  $L$  on  $\Gamma(F)$  is equivariant with respect to this  $G$ -action. Then, for any section  $\sigma$  of  $F$ , we have

$$e^{tL}(g \cdot \sigma) = g \cdot (e^{tL}\sigma).$$

This means, in terms of the heat kernel,

$$\int_M Q_t(\tilde{x}, \tilde{y}) \alpha_g \sigma(g^{-1}\tilde{y}) \text{vol}_{\tilde{y}} = \int_M \alpha_g Q_t(g^{-1}\tilde{x}, \tilde{y}) \sigma(\tilde{y}) \text{vol}_{\tilde{y}}. \quad (7.1.2)$$

Now the Riemannian volume form on  $M$  is  $G$ -invariant, provided that the metric on  $M$  is  $G$ -invariant. If this is the case, Equation 7.1.2 gives

$$\int_M Q_t(\tilde{x}, g\tilde{y}) \alpha_g \sigma(\tilde{y}) \text{vol}_{\tilde{y}} = \int_M \alpha_g Q_t(g^{-1}\tilde{x}, \tilde{y}) \sigma(\tilde{y}) \text{vol}_{\tilde{y}}. \quad (7.1.3)$$

This implies

$$Q_t(\tilde{x}, g\tilde{y}) \alpha_g = \alpha_g Q_t(g^{-1}\tilde{x}, \tilde{y}),$$

or equivalently (by taking  $\tilde{x} = \tilde{e}$  and  $g = x^{-1}$ ),

$$Q_t(\tilde{e}, x^{-1}\tilde{y}) = \alpha_x^{-1} Q_t(\tilde{x}, \tilde{y}) \alpha_x. \quad (7.1.4)$$

Hence, the heat kernel  $Q_t$  is completely determined by the function

$$q_t : \tilde{y} \mapsto Q_t(\tilde{e}, \tilde{y}).$$

We shall call  $q_t$  the *heat convolution kernel* of  $L$ . (If  $K = \{e\}$  and  $F$  is the trivial line bundle over  $G$ , then  $q_t$  is the heat convolution kernel 2.1.11 we defined in Section 2.1.9.) We may view it as a function  $]0, \infty[ \rightarrow \Gamma(F_{\tilde{e}} \boxtimes F^*)$ ,  $t \mapsto q_t$ . For small time  $t$ , the heat convolution



kernel behaves like the Gaussian<sup>1</sup> kernel

$$h_t(\bar{x}) = \frac{e^{-d(\bar{x})^2/2t}}{(2\pi t)^{\dim M/2}}, \quad (7.1.5)$$

where  $d(\bar{x})$  denotes the distance between  $\bar{e}$  and  $\bar{x}$ . More precisely,  $q_t$  admits an asymptotic expansion (or an *asymptotic heat kernel*)

$$q_t \sim h_t(a_0 + a_1 t + a_2 t^2 + \dots) \quad (7.1.6)$$

for  $t \rightarrow 0+$ , valid in the Banach<sup>2</sup> space of  $C^r$ -sections of  $F_{\bar{e}} \boxtimes F^*$  for all non-negative integer  $r$  (see [85, Thm. 7.15, p. 101]).

As it was for the asymptotic expansion for the heat kernel of the scalar Laplacian on  $G$  (see Section 2.1.9), the asymptotic expansion 7.1.6 is obtained as the formal solution to the heat equation

$$(\partial_t + L)h_t \sum_{i=0}^{\infty} a_i t^i = 0, \quad (7.1.7)$$

under the condition that  $a_0(\bar{e})$  is the identity operator on the fiber  $F_{\bar{e}}$ . Here is a sketch of the argument: Let  $\nabla$  be the connection on the vector bundle  $F$  that is associated<sup>3</sup> to  $L$ . Let  $R$  be the radial vector field in normal coordinates. Let  $\tilde{j} := j \circ \log$ , where  $j$  is the square-root of the Jacobian of the Riemannian exponential map at  $\bar{e}$ . Then, Equation 7.1.7 is equivalent to (see [85, Eq. 7.16, p. 102] or [12, Prop. 2.24, p. 82])

$$h_t \left( \partial_t + \frac{1}{t} \nabla_R + \tilde{j} \circ L \circ \frac{1}{\tilde{j}} \right) \sum_{i=0}^{\infty} a_i t^i = 0. \quad (7.1.8)$$

Setting each coefficient of the above power series in  $t$  to zero, we get the following family of differential equations that can be solved inductively:

$$\begin{aligned} \nabla_R a_0 &= 0, \\ (\nabla_R + n) a_n &= -\tilde{j} L \left( \frac{a_{n-1}}{\tilde{j}} \right), \quad n \geq 1. \end{aligned}$$

For details, see [85, Thm. 7.15, p. 101] or [12, Thm. 2.26, p. 83, Thm. 2.29, p. 85].

## 7.2 THE ASYMPTOTIC EXPANSION OF THE HEAT KERNEL OF $\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$

<sup>1</sup> The difference between Equations 2.1.16 and 7.1.5 owes to the factor of  $1/2$  present in our definition (Equation 7.1.5) of a generalized Laplacian.

<sup>2</sup> The norm is defined similarly as in Footnote 2 on page 16.

<sup>3</sup> For any generalized Laplacian  $L$  on a vector bundle  $F$ , there is a connection  $\nabla$  on  $F$  such that  $L$  is equal to (up to a zero-order differential operator)  $\Delta^F := -\sum_{i=1}^{\dim M} (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})$  where  $\{e_i\}_{i=1}^{\dim M}$  is a local orthonormal framing of  $TM$ . See [12, Prop. 2.5, p. 66].

7.2.1 SET UP. Let  $G$  be a compact connected Lie group with a bi-invariant metric. Let  $K$  be a closed connected subgroup of  $G$ , such that  $G/K$  admits a spin structure. Let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ . As we discussed in Section 6.2.8, the bi-invariance of the metric implies that the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is preserved under the action of  $\text{Ad}_k$  for all  $k$  in  $K$ , and that the restriction of  $\text{Ad}_k$  to  $\mathfrak{p}$  is in  $\text{SO}(\mathfrak{p})$ . The spin condition on  $G/K$  implies that this representation of  $K$  on  $\mathfrak{p}$  lifts to  $\text{Spin}(\mathfrak{p})$ .

$$\begin{array}{ccc} & & \text{Spin}(\mathfrak{p}) \\ & \nearrow \widetilde{\text{Ad}} & \downarrow \\ K & \xrightarrow{\text{Ad}} & \text{SO}(\mathfrak{p}) \end{array}$$

The induced Lie algebra homomorphism of  $\widetilde{\text{Ad}}$  was considered in Definition 5.4.7, and we gave it the notation

$$\gamma^{\mathfrak{p}} : \mathfrak{k} \rightarrow \mathfrak{spin}(\mathfrak{p}).$$

So we have the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\widetilde{\text{Ad}}} & \text{Spin}(\mathfrak{p}) \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{k} & \xrightarrow{\gamma^{\mathfrak{p}}} & \mathfrak{spin}(\mathfrak{p}) \end{array}$$

We consider finite-dimensional  $\mathbb{C}\text{Cl}(\mathfrak{p})$ -modules of the form

$$E = \mathbb{S} \otimes V$$

where  $\mathbb{S}$  is the space of spinors for  $\mathbb{C}\text{Cl}(\mathfrak{p})$ , and  $V$  is an irreducible  $K$ -vector space with highest weight  $\mu$ , serving as an auxiliary space on which  $\mathbb{C}\text{Cl}(\mathfrak{p})$  acts trivially. We denote the representation of  $K$  on  $V$  by

$$\pi : K \rightarrow \text{Aut}(V).$$

Then,  $K$  acts on  $E$  via

$$\mathfrak{v} := \widetilde{\text{Ad}} \otimes \pi.$$

The induced Lie algebra representation of  $\mathfrak{v}$  is the differential of  $\mathfrak{v}$  at the identity:

$$\mathfrak{v}_* = \widetilde{\text{Ad}}_* \otimes \mathbf{1} + \mathbf{1} \otimes \pi_* = \gamma^{\mathfrak{p}} + \pi_*. \quad (7.2.2)$$

Each Lie algebra representation  $\mathfrak{v}_*$ ,  $\gamma^{\mathfrak{p}}$ , and  $\pi_*$  extends to  $\mathcal{U}(\hat{\mathfrak{k}})$  as an algebra homomorphism.

7.2.3 LEMMA. As differential operators on  $(C^\infty(G) \otimes E)^K$ , we have

$$\text{diag}_{\mathcal{W}} \hat{\Omega}_{\mathfrak{k}} = -\|\mu + \rho_{\mathfrak{k}}\|^2 + \|\rho_{\mathfrak{k}}\|^2.$$

*Proof.* Let  $\sigma$  be any element of  $(C^\infty(G) \otimes E)^K$ . By the  $K$ -invariance of

$\sigma$ , we have

$$\sigma(g \exp(X)) = e^{-\nu_*(X)} \sigma(g)$$

for  $g$  in  $G$  and  $X$  in  $\mathfrak{k}$ . Differentiating both sides with respect to  $X$ , we get

$$\widehat{X}\sigma = -\nu_*(X)\sigma.$$

By Equation 7.2.2, we may rewrite this as

$$(\widehat{X} + \gamma^p(X))\sigma = -\pi_*(X)\sigma.$$

Since  $\widehat{X} + \gamma^p(X) = \text{diag}_{\mathcal{W}} \widehat{X}$  by definition (see Notation 5.4.15), we have

$$\text{diag}_{\mathcal{W}}(\widehat{X})\sigma = -\pi_*(X)\sigma.$$

Therefore,

$$\sum_{i=1}^{\dim \mathfrak{k}} \text{diag}_{\mathcal{W}}(\widehat{X}_i) \text{diag}_{\mathcal{W}}(\widehat{X}_i)\sigma = \sum_{i=1}^{\dim \mathfrak{k}} \pi_*(X_i) \pi_*(X_i)\sigma.$$

Because  $\text{diag}_{\mathcal{W}}$  and  $\pi_*$  are algebra homomorphisms, we have

$$\text{diag}_{\mathcal{W}}\left(\sum_{i=1}^{\dim \mathfrak{k}} \widehat{X}_i \widehat{X}_i\right)\sigma = \pi_*\left(\sum_{i=1}^{\dim \mathfrak{k}} X_i X_i\right)\sigma.$$

The left-hand side is  $\text{diag}_{\mathcal{W}}(\widehat{\Omega}_{\mathfrak{k}})\sigma$  and the right-hand side is  $\pi_*(\Omega_{\mathfrak{k}})\sigma$ . Hence, we have

$$\text{diag}_{\mathcal{W}}(\widehat{\Omega}_{\mathfrak{k}}) = \pi_*(\Omega_{\mathfrak{k}})$$

as differential operators on  $(C^\infty(G) \otimes \mathbb{E})^K$ . Finally, owing to the irreducibility of  $\pi$  and Schur's lemma,  $\pi_*(\Omega_{\mathfrak{k}})$  is a constant operator, whose value is

$$-\|\mu + \rho_{\mathfrak{k}}\|^2 + \|\rho_{\mathfrak{k}}\|^2;$$

see [65, Rmk. 1.89, p. 469]. This proves the lemma.  $\square$

**7.2.4 THEOREM.** *As a differential operator on  $(C^\infty(G) \otimes \mathbb{E})^K$ , the element  $\mathcal{P}(\mathfrak{g}, \mathfrak{k})^2$  is equal to*

$$\frac{1}{2}\widehat{\Omega}_{\mathfrak{g}} + \frac{1}{2}(\|\mu + \rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2). \quad (7.2.5)$$

*Proof.* By Equations 5.4.18 and 5.4.19,

$$\mathcal{P}(\mathfrak{g}, \mathfrak{k})^2 = \frac{1}{2}\widehat{\Omega}_{\mathfrak{g}} - \frac{1}{2}\text{diag}_{\mathcal{W}} \widehat{\Omega}_{\mathfrak{k}} - \frac{1}{2}\|\rho_{\mathfrak{g}}\|^2 + \frac{1}{2}\|\rho_{\mathfrak{k}}\|^2.$$

The theorem now follows from Lemma 7.2.3.  $\square$

**7.2.6** In the following,  $\hat{\Delta}_{\mathfrak{g}}$  is the element in the classical Weil algebra defined by Equation 5.3.11. It actually lies in the basic subalgebra  $S(\hat{\mathfrak{g}})^{\mathfrak{g}}$ . It is identified with the Laplacian on the Euclidean space  $\mathfrak{g}$  under the algebra isomorphism 4.1.4 with  $\Lambda = \mathfrak{g}$ .

7.2.7 THEOREM. Let  $\mathcal{Q}$  be the quantization map 5.3.8 (restricted to the  $\mathfrak{k}$ -basic subalgebra). Then

$$\mathcal{Q}(\hat{\Delta}_{\mathfrak{g}}) = \hat{\Omega}_{\mathfrak{g}} - \|\rho_{\mathfrak{g}}\|^2.$$

*Proof.* By Equation 5.3.12,

$$\frac{1}{2}\mathcal{Q}(\hat{\Delta}_{\mathfrak{g}}) = \mathcal{D}_{\mathfrak{g}}^2,$$

where  $\mathcal{D}_{\mathfrak{g}}$  is the cubic Dirac operator of  $\mathcal{W}(\mathfrak{g})$ . Meanwhile, by Equations 5.2.23 and 5.4.19,

$$\mathcal{D}_{\mathfrak{g}}^2 = \frac{1}{2}\hat{\Omega}_{\mathfrak{g}} - \frac{1}{2}\|\rho_{\mathfrak{g}}\|^2.$$

Combining the two equations above proves the theorem.  $\square$

7.2.8 DEFINITION. Let  $\widetilde{\exp}_*^{-1} : D(G) \rightarrow D(\mathfrak{g})$  be the map 4.2.16, and let  $q : \wedge(\mathfrak{p}) \rightarrow \text{Cl}(\mathfrak{p})$  be the Chevalley map 5.1.17. For a differential operator  $L$  in  $D(G) \otimes \text{Cl}(\mathfrak{p})$ , we set

$$L^{\exp} := (\widetilde{\exp}_*^{-1} \otimes q^{-1})(L).$$

*Remark.* (1) We have a vector space isomorphism,

$$\begin{array}{ccc} D(G) \otimes \text{Cl}(\mathfrak{p}) & \rightarrow & D(\mathfrak{g}) \otimes \wedge(\mathfrak{p}), \\ L & \mapsto & L^{\exp}. \end{array} \quad (7.2.9)$$

- (2) Let  $\sigma := f \otimes w$  be an element of  $C^\infty(G) \otimes E$ . Let  $\sigma^{\exp} := f^{\exp} \otimes w$ , where  $f^{\exp}$  is defined by Equation 4.2.14. Let the  $D(\mathfrak{g})$ -factor of  $L^{\exp}$  act on  $f$ , and let the  $\wedge(\mathfrak{p})$ -factor act on  $w$  via the Chevalley map. Then, by Equations 4.2.15 and 4.2.18,

$$L^{\exp} \sigma^{\exp} = (L\sigma)^{\exp}.$$

In this sense, we may understand  $L^{\exp}$  as the expression for  $L$  under a local exponential chart near the identity in  $G$ .

7.2.10 Consider, for the moment, the case where  $\mathfrak{k} = \mathfrak{g}$ . The corresponding (classical) relative Weil algebra is  $W(\mathfrak{g}, \mathfrak{g}) = S(\hat{\mathfrak{g}})^{\mathfrak{g}}$ . The element  $\hat{\Delta}_{\mathfrak{g}}$  lies in this “smallest” relative Weil algebra; and the restriction of the quantization map  $\mathcal{Q}$  to this subalgebra, as we have seen in Section 5.4.20, is the Duflo isomorphism. So it is not surprising to see the  $j$ -factor appearing in the following Proposition.

7.2.11 PROPOSITION. Let  $j$  be the function on  $\mathfrak{g}$  defined by the power series

$$j(X) = \det^{1/2} \left( \frac{\sinh \text{ad}_X / 2}{\text{ad}_X / 2} \right).$$

Then,

$$(\mathcal{Q}(\hat{\Delta}_{\mathfrak{g}}))^{\exp} = \frac{1}{j} \circ \hat{\Delta}_{\mathfrak{g}} \circ j.$$

*Remark.* The above equation is that of differential operators on  $\mathfrak{g}$ ; so  $j$  and  $1/j$  are to be understood as the multiplication by the functions  $j$  and  $1/j$ , respectively.

*Proof.* The element  $\hat{\Delta}_{\mathfrak{g}}$  lies in  $S(\hat{\mathfrak{g}})^{\mathfrak{g}}$ , which is the basic subalgebra of the classical Weil algebra  $W(\mathfrak{g})$ ; the quantization map  $\mathcal{Q}$  restricted to this subalgebra is the Duflo isomorphism (see Section 5.3.19). The assertion now follows from Proposition 4.2.20.  $\square$

**7.2.12 NOTATION.** For the rest of the Chapter,  $q_t$  denotes the heat convolution kernel of  $\mathcal{Q}(\hat{\Delta}_{\mathfrak{g}})/2$ . Then  $q_t$  is an element of  $C^\infty(G) \otimes \mathbb{C}l(\mathfrak{p})$ . We define  $q_t^{\text{exp}}$  similarly as we have done for  $\sigma^{\text{exp}}$  associated to  $\sigma$  in  $C^\infty(G) \otimes E$  (Remark (2) on page 115). We shall denote by  $h_t^{\mathfrak{g}}$  the heat convolution kernel of  $\hat{\Delta}_{\mathfrak{g}}/2$ , that is,

$$h_t^{\mathfrak{g}}(X) := \frac{e^{-\|X\|^2/2t}}{(2\pi t)^{\dim \mathfrak{g}/2}}. \quad (7.2.13)$$

**7.2.14 PROPOSITION.** *For  $t \rightarrow 0+$ , we have*

$$q_t^{\text{exp}} \sim h_t^{\mathfrak{g}} \cdot \frac{1}{j},$$

*valid in some neighborhood of 0 in  $\mathfrak{g}$ . In other words, the coefficients in the expansion 7.1.6, for this case, are  $a_0(x) = 1/j(\log(x))$  and  $a_n(x) = 0$  for  $n \geq 1$ .*

*Proof.* Let  $s_t := h_t^{\mathfrak{g}} \sum_{i=0}^{\infty} a_i t^i$  be the asymptotic expansion for  $q_t^{\text{exp}}$  (or its product with a bump function  $\psi$  such that  $\psi \equiv 1$  near 0). It is the formal solution to

$$\left( \partial_t + \frac{1}{2} \mathcal{Q}(\Delta_{\mathfrak{g}})^{\text{exp}} \right) s_t = 0.$$

We need to show that  $s_t = h_t^{\mathfrak{g}}/j$  satisfies this equation. By Proposition 7.2.11, the above is equivalent to

$$\frac{1}{j} \left( \partial_t - \frac{1}{2} \hat{\Delta}_{\mathfrak{g}} \right) j s_t = 0.$$

It is now clear that  $s_t = h_t^{\mathfrak{g}}/j$  is the solution, since  $h_t$  satisfies the heat equation  $(\partial_t - \frac{1}{2} \Delta_{\mathfrak{g}}) h_t = 0$ .  $\square$

**7.2.15 THEOREM.** *Let  $r_t$  be the heat convolution kernel of the generalized Laplacian*

$$\frac{1}{2} \hat{\Omega}_{\mathfrak{g}} + \frac{1}{2} (\|\mu + \rho_{\mathfrak{t}}\|^2 - \|\rho_{\mathfrak{g}}\|^2)$$

*on  $(C^\infty(G) \otimes \mathbb{S} \otimes V)^K$ , where  $\mathbb{S}$  is the spinor space for  $\mathbb{C}l(\mathfrak{p})$  and  $V$  is an irreducible  $K$ -vector space with highest weight  $\mu$ . Then  $r_t^{\text{exp}}$  has the following asymptotic expansion, valid in a neighborhood of 0 in  $\mathfrak{g}$ : For  $t \rightarrow 0+$ ,*

$$r_t^{\text{exp}} \sim h_t^{\mathfrak{g}} \cdot \frac{1}{j} e^{t\|\rho_{\mathfrak{t}} + \mu\|^2/2} = h_t^{\mathfrak{g}} \left( \sum_{n=0}^{\infty} \frac{1}{j} \frac{\|\rho_{\mathfrak{t}} + \mu\|^{2n}}{2^n \cdot n!} t^n \right).$$

*Proof.* By Theorem 7.2.7,

$$\frac{1}{2}\widehat{\Omega}_g + \frac{1}{2}(\|\mu + \rho_t\|^2 - \|\rho_g\|^2) = \frac{1}{2}\mathcal{Q}(\hat{\Delta}_g) + \frac{1}{2}\|\rho_t + \mu\|^2.$$

So

$$e^{t(\widehat{\Omega}_g + \|\mu + \rho_t\|^2 - \|\rho_g\|^2)/2} = e^{t\mathcal{Q}(\hat{\Delta}_g)/2} e^{t\|\rho_t + \mu\|^2/2}.$$

Hence,

$$r_t = e^{t\|\rho_t + \mu\|^2/2} q_t,$$

where  $q_t$  is the heat convolution kernel of  $\mathcal{Q}(\hat{\Delta}_g)/2$ . The asymptotic expansion for  $r_t^{\text{exp}}$  now follows from Proposition 7.2.14.  $\square$

*Remark.* Let  $q_t$  be the heat convolution kernel of  $\mathcal{Q}(\hat{\Delta}_g)/2$ . Define the linear map  $(e^{t\mathcal{Q}(\hat{\Delta}_g)/2})_e : (C^\infty(G) \otimes E)^K \rightarrow E$  by

$$(e^{t\mathcal{Q}(\hat{\Delta}_g)/2})_e \sigma = \int_G q_t(x) \sigma(x) dx.$$

Define also the linear map  $\mathcal{Q}(e^{t\hat{\Delta}_g/2})_e : (C^\infty(G) \otimes E)^K \rightarrow E$  by

$$\mathcal{Q}(e^{t\hat{\Delta}_g/2})_e \sigma = \int_G j(\log x) h_t^g(\log x) \sigma(x) \psi(x) dx,$$

where  $\psi$  is a suitable bump function on  $G$  such that

- (i) its support is contained in some neighborhood of the identity on which the inverse of the exponential map  $\log = \exp^{-1}$  is well-defined,
- (ii)  $\psi(x) = 1$  for  $x$  near the identity.

Then Proposition 7.2.14 is equivalent to the following asymptotic equality:

$$\mathcal{Q}(e^{t\hat{\Delta}_g/2})_e \sigma \sim (e^{t\mathcal{Q}(\hat{\Delta}_g)/2})_e \sigma$$

for  $t \rightarrow 0+$ . This asymptotic equality can also be proved using the (now proved) Kashiwara-Vergne conjecture [4, 60].

# 8

## THE LOCAL INDEX THEOREM ON COMPACT HOMOGENEOUS SPACES

WE now come to the primary matter — the local index theorem on a compact homogeneous space  $G/K$ . We obtained, in Chapter 7, the asymptotic expansion for the heat kernel associated to the relative Dirac operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  using the machinery of quantum Weil algebra. Now the operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  is a differential operator on  $(C^\infty(G) \otimes E)^K$  some Clifford module  $E$ ; the corresponding differential operator on  $\Gamma(G \times_K E)$  is the cubic Dirac operator  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  (Proposition 6.3.14). Our goal is to deduce the asymptotic heat kernel of  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}^2$  from that of  $\mathcal{D}^2(\mathfrak{g}, \mathfrak{k})$ , and achieve the local index theorem on  $G/K$ .

Our approach is similar to N. Berline and M. Vergne's [13] proof of the local index theorem. They considered the Riemannian Dirac operator  $\mathcal{D}$  on the vector bundle  $P_{\text{Spin}} \times_{\text{Spin}(n)} E \rightarrow M$ , where  $M$  is a generic spin manifold of dimension  $n$  and  $P_{\text{Spin}}$  is the principal  $\text{Spin}(n)$ -bundle that is a double covering of the bundle  $\text{Fr}_{\text{SO}}(M)$  of oriented orthonormal frames for the tangent bundle  $TM$  (see Section 6.2.4). They showed that the square of the Dirac operator  $\mathcal{D}$  can be identified with a second order differential operator  $Q$  on  $(C^\infty(P) \otimes E)^{\text{Spin}(n)}$  and that the operator  $Q$  is equal (up to a zero-degree operator) to a generalized Laplacian  $\tilde{\Delta}$ . They calculated the asymptotic heat kernel of  $\tilde{\Delta}$  and, from it, deduced the asymptotic heat kernel of  $\mathcal{D}^2$  on the base manifold  $M$ .

Our approach, in outline, follows the method of N. Berline and M. Vergne. What differs is that, for  $M = G/K$ , we use

- (i) the  $K$ -action on the spinors, instead of the full spin group action,

- (ii) the natural principal K-bundle  $G$  for constructing Clifford module bundles over  $G/K$ , instead of (the double covering of) the frame bundle,
- (iii) the quantum Weil algebra and the cubic Dirac operator that comes with it, rather than the Riemannian Dirac operator.

This results in a substantial simplification in the proof of the local index theorem for compact homogeneous spaces.

## 8.1 REVIEW OF THE HEAT KERNEL PROOF OF THE LOCAL INDEX THEOREM

8.1.1 We quickly recapitulate the idea behind the heat kernel proof of the local index theorem. Let  $\mathcal{D}$  be a Dirac operator on a vector bundle  $E$  over a compact, even-dimensional, spin manifold  $M$ . The space  $\Gamma(E)$  of smooth sections of  $E$  is naturally  $\mathbb{Z}/2\mathbb{Z}$ -graded (see Section 6.2.5):

$$\Gamma(E) = \Gamma(E)^+ \oplus \Gamma(E)^-.$$

The Dirac operator is an odd operator:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}. \quad (8.1.2)$$

Assuming that  $\mathcal{D}$  is symmetric<sup>1</sup> (that is, it is equal to its formal adjoint), it enjoys the following analytic properties. (For proof, see [85, Ch.5].) It is essentially self-adjoint in the  $L^2$ -closure<sup>2</sup>  $\Gamma^2(E)$  of  $\Gamma(E)$ , so we may assume that it is self-adjoint. Its spectrum form a discrete subset of  $i\mathbb{R}$ . Each eigenvalue occurs with finite multiplicity, and the eigenfunctions are smooth. The space  $\Gamma^2(E)$  of square-integrable sections of  $E$  admits a Hilbert space direct sum decomposition into the eigenspaces of  $\mathcal{D}$ .

As a consequence, the image of  $\mathcal{D}$  is equal to the orthogonal complement of the kernel of  $\mathcal{D}$ . So the usual definition of the index as the difference between the kernel and the cokernel of  $\mathcal{D}$  is trivial and uninteresting. So, when we speak of the index of a *graded* Dirac operator  $\mathcal{D}$ , we mean the following *graded index*:

$$\text{ind}_s \mathcal{D} := \dim \ker(\mathcal{D}_+) - \dim \ker(\mathcal{D}_-).$$

Now, since  $(\ker \mathcal{D})^\perp = \text{im } \mathcal{D}$ , we have  $\Gamma(E)^\pm = \ker(\mathcal{D}_\pm) \oplus \text{im}(\mathcal{D}_\mp)$ . Thus, the graded index is equal to the usual index for the operator  $\mathcal{D}_+$ ,

$$\text{ind } \mathcal{D}_+ = \dim \ker(\mathcal{D}_+) - \dim \text{coker}(\mathcal{D}_+).$$

<sup>1</sup> This is the case for geometric Dirac operators (defined by Equation 6.2.7); see [85, Prop. 3.11, p. 45]. As Equation 6.3.15 shows, the Kostant-Dirac operator is the geometric Dirac operator associated to the connection  $\nabla_X = \partial_X + \frac{1}{3}\gamma^p(X_i)$ .

<sup>2</sup> Pick a Hermitian inner  $(\cdot, \cdot)$  product for  $E \rightarrow M$  that is invariant under the Clifford action of the tangent vectors (for existence, see Section 6.2.3); and define the  $L^2$ -inner product on  $\Gamma(E)$  as  $\langle \sigma_1, \sigma_2 \rangle = [\int_M (\sigma_1(x), \sigma_2(x)) \text{vol}_x]^{1/2}$ .



8.1.3 MCKEAN-SINGER FORMULA. The kernel of  $\mathcal{D}_+$  and  $\mathcal{D}_-$  constitute the kernel of  $\mathcal{D}$ :

$$\ker \mathcal{D} = \ker \mathcal{D}_+ \oplus \ker \mathcal{D}_-.$$

The graded index of  $\mathcal{D}$  is the difference between the dimensions of the subspaces in the above decomposition. Now observe that

$$\ker \mathcal{D} = \ker \mathcal{D}^2. \quad (8.1.4)$$

This follows from the self-adjointness of  $\mathcal{D}$ , which implies

$$\langle \mathcal{D}^2 \sigma, \sigma \rangle = \langle \mathcal{D} \sigma, \mathcal{D} \sigma \rangle.$$

Thus, we have

$$\text{ind}_s \mathcal{D} = \dim \ker(\mathcal{D}^2)_+ - \dim \ker(\mathcal{D}^2)_-,$$

where  $(\mathcal{D}^2)_\pm$  denote the restriction of  $\mathcal{D}^2$  to the subspaces  $\Gamma(E)^\pm$ , respectively. Denote the  $\lambda$ -eigenspace of  $\mathcal{D}^2$  as  $\Gamma(E)_\lambda$  and let  $\Gamma(E)_\lambda^\pm := \Gamma(E)_\lambda \cap \Gamma(E)^\pm$ ; then

$$\text{ind}_s \mathcal{D} = \dim \Gamma(E)_0^+ - \dim \Gamma(E)_0^-.$$

Now we claim that, at least formally, we may add to the right-hand side the differences between the dimensions of nonzero eigenspaces:

$$\begin{aligned} \text{ind}_s \mathcal{D} &= \dim \Gamma(E)_0^+ - \dim \Gamma(E)_0^- \\ &+ \left[ \sum_{\substack{\lambda \in \text{Sp}(\mathcal{D}^2)_+ \\ \lambda \neq 0}} \dim \Gamma(E)_\lambda^+ - \sum_{\substack{\lambda \in \text{Sp}(\mathcal{D}^2)_- \\ \lambda \neq 0}} \dim \Gamma(E)_\lambda^- \right]. \end{aligned} \quad (8.1.5)$$

The rationale behind this is that there is a complete *super-symmetry* between the even and the odd nonzero eigenmodes; in other words, each nonzero eigenvalue of  $\mathcal{D}^2$  occurs with same multiplicity in the even and the odd subspaces. This is because  $\mathcal{D}$  maps an even  $\lambda$ -eigenfunction ( $\lambda \neq 0$ ) to an odd one, and vice versa; indeed, if  $\sigma$  is in  $\Gamma(E)_\lambda^\pm$ , then  $\mathcal{D}$ , as an odd operator, maps  $\sigma$  into  $\Gamma(E)_\lambda^\mp$ , and

$$\mathcal{D}^2(\mathcal{D}\sigma) = \mathcal{D}(\mathcal{D}^2\sigma) = \lambda(\mathcal{D}\sigma).$$

Now the formal equation 8.1.5 can be put into a more reasonable form if we use the operator  $e^{t\mathcal{D}^2}$ ,  $t > 0$ . This begins with the observation that

$$\text{tr } e^{t(\mathcal{D}^2)^\pm} = \dim \Gamma(E)_0^\pm + \sum_{\substack{\lambda \in \text{Sp}(\mathcal{D}^2)^\pm \\ \lambda \neq 0}} e^{t\lambda} \dim \Gamma(E)_\lambda^\pm.$$

The infinite sums are well-behaved since the multiplicity of the eigenvalues of  $\mathcal{D}^2$  is at most of polynomial growth, which is a version of Weyl's law for closed Riemannian manifolds [33]. It is also possible to directly verify that  $e^{t\mathcal{D}^2}$  is of trace-class; see [85, Thm. 8.12,

p. 114]. The point is that we can replace the heuristic equation 8.1.5 with the following well-established equation:

$$\text{ind}_s \mathcal{D} = \text{tr } e^{t(\mathcal{D}^2)_+} - \text{tr } e^{t(\mathcal{D}^2)_-} = \text{Str } e^{t\mathcal{D}^2}. \quad (8.1.6)$$

This is known as the *McKean-Singer formula*. Here  $\text{Str}$  denotes the *super-trace*; it is defined by  $\text{Str}(P) = \text{tr}(\varepsilon P)$  where  $P$  is any trace-class operator and  $\varepsilon$  is the grading operator on  $\Gamma(E)$  which takes  $\Gamma(E)^\pm$  as its  $\pm 1$ -eigenspace.

8.1.7 Here is another argument (from [12, p. 125]) for the McKean-Singer formula. We begin by expressing  $\text{Str } e^{t\mathcal{D}^2}$  in terms of the heat kernel  $Q_t$  of  $\mathcal{D}^2$ . According to a general formula for operators with smooth integral kernels (see [85, Thm. 8.12, p. 114]), we have

$$\text{Str } e^{t\mathcal{D}^2} = \int_M \text{Str}(Q_t(x, x)) \text{vol}_x. \quad (8.1.8)$$

Here  $\text{Str}(Q_t(x, x))$  is the super-trace of  $Q_t(x, x)$  as an operator on  $E_x$ , and  $\text{vol}_x$  is the Riemannian volume form at  $x$  in  $M$ . Then,

$$\frac{d}{dt} \text{Str } e^{t\mathcal{D}^2} = \int_M \text{Str}(\partial_t Q_t(x, x)) \text{vol}_x = \int_M \text{Str}(\mathcal{D}^2 Q_t(x, x)) \text{vol}_x,$$

where we have used the fact that the heat kernel satisfies the heat equation. Note that  $\mathcal{D}^2 Q_t(x, x)$  is the heat kernel of the operator  $\mathcal{D}^2 e^{t\mathcal{D}^2}$ . Thus,

$$\frac{d}{dt} \text{Str } e^{t\mathcal{D}^2} = \text{Str}(\mathcal{D}^2 e^{t\mathcal{D}^2}). \quad (8.1.9)$$

Finally, keeping in mind that  $\mathcal{D}$  commutes with  $e^{t\mathcal{D}^2}$ , we have

$$\begin{aligned} \mathcal{D}^2 e^{t\mathcal{D}^2} &= \frac{1}{2}(\mathcal{D}^2 e^{t\mathcal{D}^2} + \mathcal{D}^2 e^{t\mathcal{D}^2}) = \frac{1}{2}(\mathcal{D}^2 e^{t\mathcal{D}^2} + \mathcal{D} e^{t\mathcal{D}^2} \mathcal{D}) \\ &= \frac{1}{2}[\mathcal{D}, \mathcal{D} e^{t\mathcal{D}^2}]_s, \end{aligned} \quad (8.1.10)$$

where  $[\cdot, \cdot]_s$  is the super-commutator. Since the super-trace vanishes over super-commutators, Equations 8.1.9 and 8.1.10 imply

$$\frac{d}{dt} \text{Str } e^{t\mathcal{D}^2} = 0.$$

Thus,  $\text{Str } e^{t\mathcal{D}^2}$  is constant for  $t > 0$ . As  $t \rightarrow 0$ , the operator  $e^{t\mathcal{D}^2}$  converges to the orthogonal projection  $P_0$  onto the kernel of  $\mathcal{D}^2$ ; hence, we conclude that

$$\text{Str } e^{t\mathcal{D}^2} = \text{Str } P_0 = \text{ind}_s \mathcal{D}. \quad (8.1.11)$$

The trick of Equation 8.1.10 can be used to prove a more general result; the following proposition is from [85, Prop. 11.9, p. 144]:

8.1.12 PROPOSITION. *Let  $\phi$  be any rapidly decreasing smooth function*

on  $\mathbb{R}$  with  $\phi(0) = 1$ . Then,

$$\text{Str } \phi(\mathcal{D}^2) = \text{ind}_s \mathcal{D}.$$

*Remark.* The operator  $\phi(\mathcal{D}^2)$  is defined as the operator that acts as the scalar  $\phi(\lambda)$  on the  $\lambda$ -eigenspace of  $\mathcal{D}^2$  in  $\Gamma^2(E)$ .

*Proof.* Let  $P_0$  be the orthogonal projection onto the kernel of  $\mathcal{D}^2$ . Since the spectrum of  $\mathcal{D}^2$  is discrete, we may write  $P_0 = \psi(\mathcal{D}^2)$  for some smooth bump function  $\psi$  centered at 0 (so that  $\psi(0) = 1$  and the support of  $\psi$  is compact). Then

$$\text{ind}_s \mathcal{D} = \text{Str } P_0 = \text{Str } \psi(\mathcal{D}^2).$$

Now let  $\vartheta := \phi - \psi$ ; then

$$\text{Str } \phi(\mathcal{D}^2) = \text{Str } \vartheta(\mathcal{D}^2) + \text{Str } \psi(\mathcal{D}^2).$$

So the proposition follows once we show that  $\text{Str } \vartheta(\mathcal{D}^2) = 0$  holds for any rapidly decreasing function  $\vartheta$  with  $\vartheta(1) = 0$ . We may assume that  $\vartheta(x) = x\bar{\vartheta}(x)$  for some rapidly decreasing function  $\bar{\vartheta}$ . Then,

$$\vartheta(\mathcal{D}^2) = \mathcal{D}^2\bar{\vartheta}(\mathcal{D}^2).$$

Since  $\mathcal{D}$  commutes with  $\bar{\vartheta}(\mathcal{D}^2)$ , we may rewrite the above as

$$\vartheta(\mathcal{D}^2) = \frac{1}{2}(\mathcal{D}^2\bar{\vartheta}(\mathcal{D}^2) + \mathcal{D}\bar{\vartheta}(\mathcal{D}^2)\mathcal{D}).$$

Because  $\mathcal{D}\bar{\vartheta}(\mathcal{D}^2)$  is an odd operator, we have

$$\vartheta(\mathcal{D}^2) = \frac{1}{2}[\mathcal{D}, \mathcal{D}\bar{\vartheta}(\mathcal{D}^2)]_s.$$

Therefore,  $\text{Str } \vartheta(\mathcal{D}^2) = 0$  as desired.  $\square$

**8.1.13 LOCAL INDEX THEOREM.** By Equations 8.1.8 and 8.1.11, we have

$$\text{ind}_s(D) = \int_M \text{Str}(Q_t(x, x)) \text{vol}_x. \quad (8.1.14)$$

Because of this,  $\text{Str}(Q_t) \text{vol}$  is called the *index density* of  $\mathcal{D}$ . Since the left-hand side of this equation is independent of  $t$ , so must be the integral on the right-hand side. Examining, under this light, the small- $t$  behavior of the index density is the heat kernel approach for the Atiyah-Singer index theorem. Let

$$Q_t \sim H_t \sum_{i=0}^{\infty} A_i t^i$$

be the asymptotic expansion for  $Q_t$  as  $t \rightarrow 0+$ . Here  $H_t$  is the Gaussian kernel, and the coefficients  $A_i$  ( $i \geq 0$ ) are in  $\Gamma(E \boxtimes E^*)$ . Since

$H_t(x, x) = 1/(2\pi t)^{n/2}$ , we have

$$\text{ind}_s(D) \sim \frac{1}{(2\pi t)^{n/2}} \sum_{i=0}^{\infty} t^i \int_M \text{Str}(A_i(x, x)) \text{vol}_x. \quad (8.1.15)$$

The index theorem is achieved by showing that  $\text{Str}(A_i(x, x)) = 0$  for  $i < n/2$  and that

$$\text{Str}(A_{n/2}(x, x)) \text{vol}_x = \hat{A}(M) \wedge \text{ch}(F) \Big|_x^{\text{top}},$$

where the right-hand side denotes the top degree part of the exterior product of the Hirzebruch  $\hat{A}$ -class of  $M$  and the Chern character of the auxiliary bundle  $F := \text{End}_{\mathbb{C}\ell}(E)$ . Put in another way, the index density satisfies the asymptotic equation

$$\text{Str}(Q_t(x, x)) \text{vol}_x = \hat{A}(M) \wedge \text{ch}(F) \Big|_x^{\text{top}} + O(t),$$

for  $t \rightarrow 0+$ . This statement alone (which implies the Atiyah-Singer index theorem) is known as the *local index theorem*.

That a statement like the local index theorem might be true was suggested by H. P. McKean, Jr. and I. M. Singer [74, p. 61]. The first result in this direction is the work of V. K. Patodi [78], where he proved the Riemann-Roch-Hirzebruch theorem (which is a special case of the Atiyah-Singer index theorem) using the heat kernel method. V. K. Patodi's work was computational in nature, and it was not clear why the vanishing of the lower order coefficients in the asymptotic expansion 8.1.15 occurs. Then, P. B. Gilkey [48] showed that the cancellations of high derivatives in the computation can be explained on the grounds of Invariant Theory. Developing on P. B. Gilkey's idea, M. Atiyah, R. Bott, and V. K. Patodi [8] gave a proof for the local index theorem in its full generality; but Invariant Theory alone could not completely determine the coefficients, so they had to rely on computations of some examples. Finally, it was E. Getzler's paper [47] that demonstrated the role of Clifford algebra in the vanishing of the lower order coefficients, and gave a completely analytic proof of the Atiyah-Singer index theorem. There are other proofs of the local index theorem that appeared later on, one of which is that of N. Berline and M. Vergne [13] which we mentioned in the beginning of this chapter.

**8.1.16** Note that  $A_i(x, x)$  in the asymptotic expansion 8.1.15 takes value in  $\text{End}(E_x)$ . In the case where  $n := \dim M$  is even, we have a linear isomorphism

$$E_x \simeq \mathbb{S} \otimes V$$

where  $\mathbb{S}$  is the spinor space for  $\mathbb{C}\ell(n)$  and  $V$  is some auxiliary vector space (see Section 5.1.5). The  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $E_x$  arises from that of  $\mathbb{S}$ . An element of  $\text{End}(E_x)$  can be identified with an element  $\sum w \otimes F$

in  $\mathbb{Cl}(n) \otimes \text{End}(V)$ . And

$$\text{Str}(w \otimes F) = \text{tr}_s(w) \text{tr}_V(F)$$

where  $\text{tr}_s(w)$  is the super-trace of the action of  $w$  on  $\mathbb{S}$  and  $\text{tr}_V(F)$  is the usual trace of  $F$  over  $V$ .

We would like to mention at this point that the super-trace  $\text{tr}_s(w)$  is nonzero only if  $w$  contains a term that has top filtration order in  $\mathbb{Cl}(n)$ . To be more precise, let  $e_1, \dots, e_n$  be an orthonormal basis for  $\mathbb{R}^n$ ; then,

$$\text{tr}_s(e_I) = \begin{cases} (-i)^{n/2}, & \text{if } I = \{1, 2, \dots, n\}, \\ 0, & \text{otherwise,} \end{cases} \quad (8.1.17)$$

where  $e_I$  are the elements in  $\mathbb{Cl}(n)$  defined by Equation 5.1.9. To see why this is true, note first that the super-trace vanishes on super-commutators. If  $I \neq \{1, \dots, n\}$ , then  $e_I$  is a super-commutator:  $e_I = -[e_I e_i, e_i]_s$  for any  $i$  not in  $I$ . To calculate  $\text{tr}_s(e_I)$  for  $I = \{1, \dots, n\}$ , we use the fact that the space of even and odd spinors are the  $\pm 1$ -eigenspaces of the action of  $\omega := (2i)^{n/2} e_1 \cdots e_n$  (see Section 5.1.44). Hence,

$$\text{tr}_s(e_1 \cdots e_n) = \text{tr}(e_1 \cdots e_n \omega) = (2i)^{n/2} \text{tr}((e_1 \cdots e_n)^2).$$

Since  $e_i e_j = -e_j e_i$  for  $i \neq j$ ,

$$(e_1 \cdots e_n)^2 = (-1)^{n(n-1)/2} e_1^2 \cdots e_n^2.$$

Because  $(-1)^{n(n-1)/2} = (-1)^{n/2}$  and  $e_i^2 = 1/2$ , we have

$$(e_1 \cdots e_n)^2 = (-1)^{n/2} 2^{-n}.$$

Thus,

$$\text{tr}_s(e_1 \cdots e_n) = (-i)^{n/2} 2^{-n/2} \dim(\mathbb{S}).$$

Since  $\dim(\mathbb{S}) = 2^{n/2}$  (see Section 5.1.5),

$$\text{tr}_s(e_1 \cdots e_n) = (-i)^{n/2}.$$

## 8.2 THE LOCAL INDEX THEOREM ON $G/K$

**8.2.1 NOTATIONS.** We bring in the notations that we set up in Section 7.2.1. Thus,  $G$  is a compact connected Lie group with a bi-invariant metric, and  $K$  is a closed connected subgroup of  $G$ . We have a Lie group homomorphism

$$\widetilde{\text{Ad}}: K \rightarrow \text{Spin}(\mathfrak{p})$$

where  $\mathfrak{p}$  is the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ . The induced Lie algebra homomorphism is

$$\gamma^{\mathfrak{p}}: \mathfrak{k} \rightarrow \mathfrak{spin}(\mathfrak{p}).$$

A formula for  $\gamma^p$  is given by Equation 5.4.10. We have a twisted  $\text{Cl}(\mathfrak{p})$ -module

$$E = \mathbb{S} \otimes V$$

where  $\mathbb{S}$  is the space of spinors for  $\text{Cl}(\mathfrak{p})$  and the auxiliary space  $V$  is an irreducible  $K$ -vector space with highest weight  $\mu$ . We denote the representation of  $K$  on  $V$  by

$$\pi: K \rightarrow \text{Aut}(V).$$

Thus,  $E$  is a  $K$ -vector space via

$$\nu := \widetilde{\text{Ad}} \otimes \pi.$$

The induced Lie algebra representation of  $\nu$  is the differential of  $\nu$  at the identity:

$$\nu_* = \widetilde{\text{Ad}}_* \otimes 1 + 1 \otimes \pi_* = \gamma^p + \pi_*. \quad (8.2.2)$$

Each Lie algebra representation  $\nu_*$ ,  $\gamma^p$ , and  $\pi_*$  extends to  $\mathcal{U}(\mathfrak{k})$  as an algebra homomorphism.

Let us add few more notations that we shall use. For the rest of this chapter,  $\{X_i\}_{i=1}^{\dim \mathfrak{k}}$  and  $\{Y_i\}_{i=1}^{\dim \mathfrak{p}}$  denote any orthonormal basis for  $\mathfrak{k}$  and  $\mathfrak{p}$ , respectively. For each vector  $X$  in  $\mathfrak{g}$ , we denote by  $X^*$  the linear functional on  $\mathfrak{g}$  defined by

$$X^*(Y) = \langle X, Y \rangle$$

for  $Y$  in  $\mathfrak{g}$ . Hence,  $\{X_i^*\}_{i=1}^{\dim \mathfrak{k}}$  and  $\{Y_i^*\}_{i=1}^{\dim \mathfrak{p}}$  serves as basis for  $\mathfrak{k}^*$  and  $\mathfrak{p}^*$ , respectively.

We shall denote by  $\lambda^p$  the composition

$$\mathfrak{k} \xrightarrow{\gamma^p} \mathfrak{spin}(\mathfrak{p}) \xrightarrow{q^{-1}} \wedge^2 \mathfrak{p}$$

where  $q^{-1}$  is the inverse of the Chevalley map 5.1.17. By Equation 5.4.10, we have

$$\lambda^p(X) = -\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{p}} \langle X, [Y_i, Y_j]_{\mathfrak{g}} \rangle Y_i Y_j. \quad (8.2.3)$$

We shall often use the notation

$$\bar{x} := xK$$

for the points in  $G/K$ .

Finally, recall that our model for the tangent bundle  $T(G/K)$  is  $T \times_K \mathfrak{p}$  (see Section 6.2.8). This means that we are identifying the tangent space  $T_{\bar{e}}(G/K)$  and the cotangent space  $T_{\bar{e}}^*(G/K)$  with  $\mathfrak{p}$  and  $\mathfrak{p}^*$ , respectively.

**8.2.4** As far as index theory is concerned, R. Bott showed that a nontrivial index can occur only if  $K$  is of maximal rank (that is,  $K$  contains a maximal torus of  $G$ ) [16, Thm. II, p. 170]. So we shall concentrate on the case where  $K$  is of maximal rank. Tow implica-

tions of this are:

- (i) The dimension of  $G/K$ , which is equal to that of  $\mathfrak{p}$ , is even. To see this, pick a common maximal torus for  $G$  and  $K$  so that their Lie algebras share the same maximal commutative subalgebra in their root space decomposition. Let  $\Phi_{\mathfrak{g}}$  and  $\Phi_{\mathfrak{k}}$  be the set of roots of  $G$  and  $K$ , respectively. Then  $\Phi_{\mathfrak{k}}$  is a subset of  $\Phi_{\mathfrak{g}}$ , and the dimension of  $\mathfrak{p}$  is equal to the cardinality of  $\Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{k}}$ , which is even, since the roots come in positive and negative pairs (see Section 2.3.11).
- (ii) Because  $\dim(\mathfrak{p})$  is even, the spinor space  $\mathbb{S}$  for  $\text{Cl}(\mathfrak{p})$  is naturally  $\mathbb{Z}/2\mathbb{Z}$ -graded (see Section 5.1.44). This gives rise to  $\mathbb{Z}/2\mathbb{Z}$ -gradings on the Clifford module  $E$ , the associated vector bundle  $G \times_{\vee} E$ , and the space of smooth sections of  $G \times_{\vee} E$ .

*Example (The Flag Manifold  $G/T$ ).* As a special case, suppose  $K$  is equal to a maximal torus  $T$  of  $G$ . Then  $G/T$  is  $\text{spin}$  [41, Cor.1.12, p. 91]. The homomorphism

$$\widetilde{\text{Ad}} : T \rightarrow \text{Spin}(\mathfrak{p}) \quad (8.2.5)$$

gives rise to a  $T$ -action on the spinor space  $\mathbb{S}$ . The complexification of  $\mathfrak{p}$  is the union of the root spaces of  $\mathfrak{g}$ . Let  $\Phi^+ = \{\alpha_1, \dots, \alpha_k\}$  be the selected set of positive roots of  $\mathfrak{g}$ . (Hence,  $k = \dim \mathfrak{p}/2$ .) As we shall soon see, the set of the weights of the  $T$ -representation on  $\mathbb{S}$  is

$$\left\{ \frac{1}{2}(\beta_1 + \dots + \beta_k) : (\beta_1, \dots, \beta_k) \in \prod_{i=1}^k \{\pm \alpha_i\} \right\}, \quad (8.2.6)$$

the highest weight being  $\rho := \frac{1}{2} \sum_{i=1}^k \alpha_i$ ; for the multiplicities, the  $k$ -tuple  $(\beta_1, \dots, \beta_k)$  contributes with multiplicity 1 to the weight  $\frac{1}{2}(\beta_1 + \dots + \beta_k)$ . So  $\dim(\mathbb{S}) = 2^k$ , which is consistent with the general theory of spinors.

The weights and their multiplicities can be verified as follows. (The line of argument is from [68, p. 11].) The complexification of  $\mathfrak{p}$  is simply the collection of all the root spaces:

$$\mathfrak{p}_{\mathbb{C}} = \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_{\mathbb{C}, \alpha} \oplus \mathfrak{g}_{\mathbb{C}, -\alpha}).$$

Choose vectors  $Y_{\alpha}$  in  $\mathfrak{g}_{\mathbb{C}, \alpha}$  for each positive root  $\alpha$ , and let  $Y_{-\alpha}$  be the vector such that  $\langle Y_{\alpha}, Y_{-\alpha} \rangle = 1$ ; since the root spaces  $\mathfrak{g}_{\mathbb{C}, \alpha}$  and  $\mathfrak{g}_{\mathbb{C}, \beta}$  are orthogonal if  $\alpha \neq \pm \beta$ , the vector  $Y_{-\alpha}$  is necessarily in  $\mathfrak{g}_{\mathbb{C}, -\alpha}$ . Their collection  $\{Y_{\pm \alpha}\}_{\alpha \in \Phi^+}$  form a basis for  $\mathfrak{p}_{\mathbb{C}}$ . To find the weights of the  $T$ -representation on  $\mathbb{S}$ , we need to examine the induced Lie algebra representation, which is given by  $\gamma^{\mathfrak{p}} : \mathfrak{t} \rightarrow \mathfrak{spin}(\mathfrak{p})$ . By Equation 5.4.10,

$$\gamma^{\mathfrak{p}}(H) = \frac{1}{2} \sum_{\alpha \in \Phi^+} (\text{ad}_H(Y_{\alpha})Y_{-\alpha} + \text{ad}_H(Y_{-\alpha})Y_{\alpha}).$$

Since  $Y_{\pm\alpha}$  are root vectors, satisfying  $\text{ad}_H(Y_{\pm\alpha}) = \pm\alpha(H)Y_{\pm\alpha}$ ,

$$\gamma^p(H) = \frac{1}{2} \sum_{\alpha \in \Phi^+} i\alpha(H)(Y_\alpha Y_{-\alpha} - Y_{-\alpha} Y_\alpha).$$

Since  $Y_\alpha Y_{-\alpha} = -Y_{-\alpha} Y_\alpha + \langle Y_\alpha, Y_{-\alpha} \rangle$  in  $\text{Cl}(\mathfrak{p})$  and  $\langle Y_\alpha, Y_{-\alpha} \rangle = 1$ ,

$$\begin{aligned} \gamma^p(H) &= \frac{1}{2} \sum_{\alpha \in \Phi^+} i\alpha(H) - \sum_{\alpha \in \Phi^+} i\alpha(H)Y_{-\alpha}Y_\alpha \\ &= i\rho(H) - \sum_{\alpha \in \Phi^+} i\alpha(H)Y_{-\alpha}Y_\alpha. \end{aligned} \quad (8.2.7)$$

The Clifford relation can be used in the other way around to get

$$\gamma^p(H) = -i\rho(H) + \sum_{\alpha \in \Phi^+} i\alpha(H)Y_\alpha Y_{-\alpha}. \quad (8.2.8)$$

Now, if  $v$  is a weight vector in  $\mathbb{S}$  with weight  $\lambda$ , then the action of  $Y_{\pm\alpha}$  maps  $v$  to a weight vector with weight  $\lambda \pm \alpha$ ; this owes to the following super-commutator relation:

$$[\gamma^p(H), Y_{\pm\alpha}] = \text{ad}_H(Y_{\pm\alpha}) = \pm i\alpha(H)Y_\alpha.$$

(The first equality is just Equation 5.1.54.) Now suppose  $v$  is a weight vector with highest weight. Then  $Y_\alpha \cdot v$  must be zero, and hence, by Equation 8.2.7,

$$\gamma^p(H) \cdot v = i\rho(H)v.$$

Therefore,  $\rho$  is the weight of  $v$ , which is the highest weight of the  $T$ -action on  $\mathbb{S}$ . Similar argument using Equation 8.2.8 tells us that the lowest weight is  $-\rho$ , which is equal to  $\rho - \sum_{\alpha \in \Phi^+} \alpha$ . Hence,  $\rho - \sum_{\alpha \in I} \alpha$  for  $I \subseteq \Phi^+$  gives us all the weights, each  $I$  contributing with multiplicity 1.

8.2.9 Let

$$\Xi: \Gamma(G \times_\nu E) \xrightarrow{\sim} (C^\infty(G) \otimes E)^K$$

be the vector space isomorphism defined in Section 6.3.1. By Proposition 6.3.14,  $\Xi$  intertwines the differential operators  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  on  $\Gamma(G \times_\mu E)$  and  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  on  $(C^\infty(G) \otimes E)^K$ . Now, by Theorem 7.2.4,  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})^2$  is equal to

$$L := \frac{1}{2} \widehat{\Omega}_{\mathfrak{g}} + \frac{1}{2} (\|\mu + \rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2)$$

as operators on  $(C^\infty(G) \otimes E)^K$ . Hence,  $\Xi$  intertwines  $e^{t\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}^2}$  and  $e^{tL}$ .

$$\begin{array}{ccc} (C^\infty(G) \otimes E)^K & \xrightarrow{e^{tL}} & (C^\infty(G) \otimes E)^K \\ \Xi \uparrow \wr & & \wr \uparrow \Xi \\ \Gamma(G \times_\nu E) & \xrightarrow{e^{t\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}}} & \Gamma(G \times_\nu E) \end{array} \quad (8.2.10)$$



We wish to compare the heat kernel of  $\mathcal{D}_{g/\mathfrak{k}}^2$  with that of  $L$ . To that end, recall that a 2-tuple  $(\bar{\sigma}, \sigma)$  in  $\Gamma(G \times_{\mathfrak{v}} E) \times (C^\infty(G) \otimes E)^K$  is in the graph of  $\Xi$  if and only if

$$\bar{\sigma}(\bar{g}) = [g, \sigma(g)] \quad (8.2.11)$$

for any  $g$  in  $G$ . With this in mind, we adopt the following definition:

**8.2.12 DEFINITION.** Let  $Q_t$  ( $t > 0$ ) be an integral kernel for some operator on  $\Gamma(G \times_{\mathfrak{v}} E)$ , where  $G/K$  is endowed with the quotient measure (see Section 2.3.24). Consider a function

$$\tilde{Q}_t : G \times G \rightarrow \text{End}(E)$$

that satisfies the following two conditions:

(i) For all  $(k_1, k_2)$  in  $K \times K$ ,

$$\tilde{Q}_t(xk_1, yk_2) = \mathfrak{v}(k_1)^{-1} \tilde{Q}_t(x, y) \mathfrak{v}(k_2). \quad (8.2.13)$$

(ii) For any  $(\bar{\sigma}, \sigma)$  in the graph of  $\Xi$ ,

$$\int_{G/K} Q_t(\bar{x}, \bar{y}) \bar{\sigma}(\bar{y}) d\bar{y} = \left[ x, \int_{G/K} \tilde{Q}_t(x, y) \sigma(y) dy \right]. \quad (8.2.14)$$

(In the above,  $[g, v]$  is the notation for the  $K$ -orbit of  $(g, v)$  in  $G \times E$ .)

Condition (i) is necessary for Equation 8.2.14 to make sense. If such  $\tilde{Q}_t$  exists, we shall call it the kernel on  $G$  that is *equivalent* to  $Q_t$ .

**8.2.15 LEMMA.** Let  $R_t$  be the heat kernel of  $\frac{1}{2}\hat{\Omega}_g + \frac{1}{2}(\|\mu + \rho_{\mathfrak{k}}\|^2 - \|\rho_g\|^2)$  on  $(C^\infty(G) \otimes E)^K$ .

$$R_t(xk_1, yk_2) = \mathfrak{v}(k_1)^{-1} R_t(x, y) \mathfrak{v}(k_2)$$

for  $(x, y)$  in  $G \times G$  and  $(k_1, k_2)$  in  $K \times K$ .

*Remark.* A version of this can be found in [13, Eq. 3.18, p. 333].

*Proof.* Let  $\sigma$  be an arbitrary element in  $(C^\infty(G) \otimes E)^K$ . Then

$$\sigma(yk) = \mathfrak{v}(k)^{-1} \sigma(y)$$

for  $y$  in  $G$  and  $k$  in  $K$ . Thus,

$$\begin{aligned} \int_G R_t(x, yk) \mathfrak{v}(k)^{-1} \sigma(y) dy &= \int_G R_t(x, yk) \sigma(yk) dy \\ &= \int_G R_t(x, y) \sigma(y) d(yk^{-1}) = \int_G R_t(x, y) \sigma(y) dy. \end{aligned}$$

This proves that

$$R_t(x, yk) \mathfrak{v}(k)^{-1} = R_t(x, y).$$

Next, since the map

$$x \mapsto \int_G R_t(x, y) \sigma(y) dy$$

defines an element of  $(C^\infty(G) \otimes E)^K$ , we have

$$\int_G R_t(xk, y) \sigma(y) dy = \int_G v(k)^{-1} R_t(x, y) \sigma(y) dy.$$

This proves that

$$R_t(xk, y) = v(k)^{-1} R_t(x, y). \quad \square$$

**8.2.16 LEMMA.** *Let  $Q_t$  be the heat kernel of  $\mathcal{D}_{g/\mathfrak{e}}^2$  on  $\Gamma(G \times_v E)$ . Let  $R_t$  be the heat kernel of  $\frac{1}{2}\widehat{\Omega}_g + \frac{1}{2}(\|\mu + \rho_{\mathfrak{e}}\|^2 - \|\rho_g\|^2)$  on  $(C^\infty(G) \otimes E)^K$ . Then,*

$$\tilde{Q}_t(x, y) := \int_K R_t(x, yk) v(k)^{-1} dk \quad (8.2.17)$$

*defines an integral kernel on  $G$  that is equivalent to  $Q_t$ .*

*Remark.* A version of this can be found in [13, Prop. 3.20, p. 333].

*Proof.* Let  $k_1$  and  $k_2$  be arbitrary elements of  $K$ . By Lemma 8.2.15 and the invariance of the measure on  $K$ , we have

$$\begin{aligned} \tilde{Q}_t(xk_1, yk_2) &= \int_K R_t(xk_1, yk_2k) v(k)^{-1} dk \\ &= \int_K v(k_1)^{-1} R_t(x, yk) v(k_2^{-1}k)^{-1} d(k_2^{-1}k) \\ &= \int_K v(k_1)^{-1} R_t(x, yk) v(k)^{-1} v(k_2) dk \\ &= v(k_1)^{-1} Q_t(x, y) v(k_2). \end{aligned}$$

This checks property 8.2.13.

It remains to check Equation 8.2.14. Let  $(\bar{\sigma}, \sigma)$  be an arbitrary 2-tuple in the graph of  $\Xi$ . Let  $x$  be an arbitrary point in  $G$ . Suppose

$$(e^{t\mathcal{D}_{g/\mathfrak{e}}^2} \bar{\sigma})(\bar{x}) = [x, w]. \quad (8.2.18)$$

We need to show that

$$w = \int_{G/K} \tilde{Q}_t(x, y) \sigma(y) d\bar{y}. \quad (8.2.19)$$

Since  $(e^{t\mathcal{D}_{g/\mathfrak{e}}^2} \bar{\sigma}, e^{t(\widehat{\Omega}_g + \|\mu + \rho_{\mathfrak{e}}\|^2 - \|\rho_g\|^2)/2} \sigma)$  is in the graph of  $\Xi$ , Equations 8.2.11 and 8.2.18 imply

$$w = (e^{t(\widehat{\Omega}_g + \|\mu + \rho_{\mathfrak{e}}\|^2 - \|\rho_g\|^2)/2} \sigma)(x).$$

This can be rewritten in terms of the heat kernel as

$$w = \int_G R_t(x, y) \sigma(y) dy.$$

Then, by the fact that

$$\int_G f(g) dg = \int_{G/K} \int_K f(gk) dk d\bar{g}$$

for any continuous function  $f$  on  $G$  (see [34, Eq. 3.13.19, p.184]),

we have

$$w = \int_{G/K} \int_K R_t(x, yk) \sigma(yk) dk d\bar{y}.$$

Since  $\sigma(yk) = \nu(k)^{-1} \sigma(y)$ ,

$$\begin{aligned} w &= \int_{G/K} \int_K R_t(x, yk) \nu(k)^{-1} \sigma(y) d\bar{y} \\ &= \int_{G/K} \tilde{Q}_t(x, y) \sigma(y) d\bar{y}. \end{aligned}$$

This is Equation 8.2.19, which proves that  $\tilde{Q}_t$  satisfies Equation 8.2.14.  $\square$

8.2.20 COROLLARY. Let  $\tilde{Q}_t$  and  $R_t$  be as in Lemma 8.2.16. Let  $\tilde{q}_t$  and  $r_t$  be the convolution kernels associated to  $\tilde{Q}_t$  and  $R_t$ , respectively. Then

$$\tilde{q}_t(g) = \int_K r_t(gk) \nu(k)^{-1} dk$$

for any  $g$  in  $G$ .

*Proof.* By definition (see Section 7.1),

$$\begin{aligned} \tilde{q}_t(g) &= \tilde{Q}_t(e, g), \\ r_t(g) &= R_t(e, g). \end{aligned}$$

So, by Lemma 8.2.16,

$$\tilde{q}_t(g) = \int_K R_t(e, gk) \nu(k)^{-1} dk = \int_K r_t(gk) \nu(k)^{-1} dk. \quad \square$$

8.2.21 NOTATION. Recall that the orthogonal decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is an  $\text{ad}(\mathfrak{k})$ -invariant decomposition. For each  $X$  in  $\mathfrak{k}$ , let  $\text{ad}_X^{\mathfrak{k}}$  and  $\text{ad}_X^{\mathfrak{p}}$  denote the restrictions of  $\text{ad}_X$  to  $\mathfrak{k}$  and  $\mathfrak{p}$ , respectively. Then the matrix for  $\text{ad}_X$  is a block diagonal matrix of the form  $\begin{pmatrix} \text{ad}_X^{\mathfrak{k}} & 0 \\ 0 & \text{ad}_X^{\mathfrak{p}} \end{pmatrix}$ . Hence the power series

$$j(X) = \det^{1/2} \left( \frac{\sinh(\text{ad}_X/2)}{\text{ad}_X/2} \right)$$

factors into

$$j(X) = j_{\mathfrak{k}}(X) j_{\mathfrak{g}/\mathfrak{k}}(X) \quad (8.2.22)$$

where  $j_{\mathfrak{k}}$  and  $j_{\mathfrak{g}/\mathfrak{k}}$  are defined by

$$j_{\mathfrak{k}}(X) := \det^{1/2} \left( \frac{\sinh(\text{ad}_X^{\mathfrak{k}}/2)}{\text{ad}_X^{\mathfrak{k}}/2} \right), \quad j_{\mathfrak{g}/\mathfrak{k}}(X) := \det^{1/2} \left( \frac{\sinh(\text{ad}_X^{\mathfrak{p}}/2)}{\text{ad}_X^{\mathfrak{p}}/2} \right). \quad (8.2.23)$$

8.2.24 LEMMA. Let  $\tilde{q}_t$  be as in Corollary 8.2.20. For  $t \rightarrow 0+$ , we have an asymptotic equality

$$\tilde{q}_t(e) \sim e^{t\|\rho_{\mathfrak{k}} + \mu\|^2/2} \int_{\mathfrak{k}} \frac{j_{\mathfrak{k}}(X)}{j_{\mathfrak{g}/\mathfrak{k}}(X)} h_t^{\mathfrak{g}}(X) e^{-\nu_*(X)} dX$$

as  $\text{End}(E)$ -valued functions of  $t$ .

*Proof.* By Theorem 7.2.15 and Corollary 8.2.20, we have

$$\tilde{q}_t(e) \sim e^{t\|\rho_{\mathfrak{k}}+\mu\|^2/2} \int_{\mathfrak{k}} \frac{j_{\mathfrak{k}}(X)^2}{j(X)} h_t^g(X) e^{-v_*(X)} dX.$$

The assertion then follows from Equation 8.2.22.  $\square$

The following lemma is from [35, Lem. 11.3, p. 137]:

**8.2.25 LEMMA.** *Let  $\varphi$  be a smooth function on  $\mathfrak{k}$  with sufficiently slow growth. Then, the  $\wedge(\mathfrak{p})$ -valued function*

$$t \mapsto \int_{\mathfrak{k}} h_t^{\mathfrak{k}}(X) \varphi(X) e^{-\lambda^{\mathfrak{p}}(X)} dX$$

*has an asymptotic expansion  $\sum_{n=0}^{\infty} t^n \Psi_n$  for  $t \rightarrow 0+$ . The  $n$ th coefficient  $\Psi_n$  is contained in  $\bigoplus_{\ell=0}^n \wedge^{2\ell}(\mathfrak{p})$ . If  $n \leq \dim(\mathfrak{p})/2$ , then the component of  $\Psi_n$  in  $\wedge^{2n}(\mathfrak{p})$  (the highest degree part) is given by*

$$\Psi_n^{(2n)} = \frac{1}{n!} \left( \sum_{i=1}^{\dim \mathfrak{k}} -\lambda^{\mathfrak{p}}(X_i) \partial_i \right)^n \varphi(X) \Big|_{X=0}, \quad (8.2.26)$$

where  $\partial_i$  is the partial derivative with respect to the  $i$ th coordinate variable  $x_i : X \mapsto \langle X_i, X \rangle$ .

*Proof.* The Gaussian function  $h_t^{\mathfrak{k}}$  is the heat convolution kernel of the generalized Laplacian  $\hat{\Delta}_{\mathfrak{k}}/2$ . Thus, for any smooth function  $\psi$  on  $\mathfrak{k}$  with sufficiently slow growth,

$$(e^{t\hat{\Delta}_{\mathfrak{k}}/2}\psi)(0) = \int_{\mathfrak{k}} h_t^{\mathfrak{k}}(X) \psi(X) dX. \quad (8.2.27)$$

This is asymptotically equal to  $\sum_{n=0}^{\infty} \frac{t^n}{n!} (\frac{1}{2^n} \hat{\Delta}_{\mathfrak{k}}^n \psi)(0)$  as  $t \rightarrow 0+$ . (This is fairly easy to check by substituting  $\psi$  with its Taylor series on the right-hand side of Equation 8.2.27 and applying the formula  $\int_{\mathbb{R}} e^{-x^2/2} x^{2k} dx = \sqrt{2\pi} (2k)!/(k!)$ ; see [12, Prop. 2.13, p. 73] for details.) Hence,

$$\int_{\mathfrak{k}} h_t^{\mathfrak{k}}(X) \varphi(X) e^{-\lambda(X)} dX \sim \sum_{n=0}^{\infty} t^n \Psi_n,$$

where

$$\Psi_n := \frac{1}{n!} \left( \frac{\hat{\Delta}_{\mathfrak{k}}}{2} \right)^n (\varphi(X) e^{-\lambda(X)}) \Big|_{X=0}.$$

Now

$$\begin{aligned}
\frac{\hat{\Delta}_{\mathfrak{k}}}{2}(\varphi(X)e^{-\lambda(X)}) &= \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{k}} \hat{X}_i \hat{X}_i (\varphi(X)e^{-\lambda(X)}) \\
&= \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{k}} \left( \frac{1}{2} \partial_i^2 \varphi(X) e^{-\lambda(X)} + 2 \partial_i \varphi(X) \partial_i e^{-\lambda(X)} + \varphi(X) \partial_i^2 e^{-\lambda(X)} \right) \\
&= \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{k}} \left( \partial_i^2 \varphi(X) - 2 \partial_i \varphi(X) \lambda(X_i) + 4 \varphi(X) \lambda(X_i) \lambda(X_i) \right).
\end{aligned}$$

The last term is zero because  $\lambda(X_i) \lambda(X_i) = 0$ . Thus,

$$\frac{\hat{\Delta}_{\mathfrak{k}}}{2}(\varphi(X)e^{-\lambda(X)}) = \left[ \left( \sum_{i=1}^{\dim \mathfrak{k}} \frac{1}{2} \partial_i^2 - \lambda^{\mathfrak{p}}(X_i) \partial_i \right) \varphi(X) \right] e^{-\lambda(X)}.$$

Therefore,

$$\left( \frac{\hat{\Delta}_{\mathfrak{k}}}{2} \right)^n (\varphi(X) e^{-\lambda(X)}) = \left[ \left( \sum_{i=1}^{\dim \mathfrak{k}} \frac{1}{2} \partial_i^2 - \lambda^{\mathfrak{p}}(X_i) \partial_i \right)^n \varphi(X) \right] e^{-\lambda(X)},$$

and hence,

$$\Psi_n = \frac{1}{n!} \left( \sum_{i=1}^{\dim \mathfrak{k}} \frac{1}{2} \partial_i^2 - \lambda^{\mathfrak{p}}(X_i) \partial_i \right)^n \varphi(X) \Big|_{X=0}.$$

Since  $\lambda(X_i)$  is in  $\wedge^2(\mathfrak{p})$ , we see that the exterior algebra factor of the terms in  $\Psi_n$  are all of even degree at most  $2n$ . And, if  $2n \leq \dim(\mathfrak{p})$ , the component of  $\Psi_n$  that has degree  $2n$  is given by Equation 8.2.26.  $\square$

8.2.28 Let  $\varphi$  be as in Lemma 8.2.25. The  $n$ th order term of the Taylor series for  $\varphi(X)$  with respect to  $X = 0$  is

$$\frac{1}{n!} \left( \sum_{i=1}^{\dim \mathfrak{k}} x_i \partial_i \right)^n \varphi(X) \Big|_{X=0}.$$

As observed by N. Berline and M. Vergne [13, § 1.23, p. 314], substituting  $x_i$  with  $-\lambda^{\mathfrak{p}}(X_i)$  in the above expression gives the right-hand side of Equation 8.2.26. Now the coordinate variable  $x_i$  is just the linear functional  $X_i^*$ . Hence, the Taylor series yields an algebra homomorphism  $C^\infty(\mathfrak{k}) \rightarrow \mathbb{R}[[\mathfrak{k}^*]]$ , where the codomain is the algebra of formal power series in  $\mathfrak{k}^*$  over  $\mathbb{R}$ ; composing this with the algebra homomorphism  $\mathbb{R}[[\mathfrak{k}^*]] \rightarrow \wedge(\mathfrak{p}^*)$  generated by  $X^* \mapsto \wedge^{\mathfrak{p}}(X)$  yields an algebra homomorphism

$$C^\infty(\mathfrak{k}) \rightarrow \wedge(\mathfrak{p}^*). \tag{8.2.29}$$

Note that this homomorphism factors through the symmetric algebra  $S(\mathfrak{k}^*)$ .

8.2.30 NOTATION. Let  $f$  be a smooth function on  $\mathfrak{k}$ . We shall denote

its image under the homomorphism 8.2.29 by  $f(-\lambda^p X)$ .

**8.2.31 DEFINITION.** Let  $p$  be the projection map of the principal  $K$ -bundle  $G \rightarrow G/K$ . Let  $U$  be a locally trivializing neighborhood containing the identity coset  $\bar{e}$  in  $G/K$ ; let  $\xi : p^{-1}(U) \rightarrow U \times K$  be the trivialization over  $U$ . Consider the composition map

$$p^{-1}(U) \xrightarrow{\xi} U \times K \rightarrow K,$$

where the second map is the projection onto the second component. We shall denote by  $\theta$  the pullback of the Maurer-Cartan connection on  $K$  along the above composition, and call it the *local Maurer-Cartan connection* near the identity in the principal  $K$ -bundle  $G$  (see the example on page 94). At the identity,  $\theta$  is the  $\mathfrak{k}$ -valued 1-form

$$\theta = \sum_{i=1}^{\dim \mathfrak{k}} X_i \otimes X_i^* \in \mathfrak{k} \otimes \mathfrak{g}^*.$$

The curvature of  $\theta$  shall be denoted by  $\Theta$ . At the identity,  $\Theta$  is of the form

$$\Theta = \sum_{i=1}^{\dim \mathfrak{k}} X_i \otimes \Theta^i \in \mathfrak{k} \otimes \wedge^2(\mathfrak{g}^*),$$

where  $\Theta^i$  is a horizontal 2-form. Note that the horizontal subspace of  $\mathfrak{g}$  with respect to  $\theta$  is  $\mathfrak{p}$ . Hence, each  $\Theta^i$  is contained in the subalgebra  $\wedge^2(\mathfrak{p}^*)$  of  $\wedge^2(\mathfrak{g}^*)$ .

**8.2.32 NOTATION.** The local Maurer-Cartan connection  $\theta$  on the principal  $K$ -bundle  $G \rightarrow G/K$  induces local connections for the tangent bundle  $T \times_K \mathfrak{p}$  and the twisting bundle  $G \times_K V$  over  $G/K$  (see Section 6.1.29). We shall denote their curvature 2-forms, respectively, by  $\Theta_T$  and  $\Theta_V$ ; at the identity coset  $\bar{e}$ , they take the following form:

$$\Theta_T = - \sum_{i=1}^{\dim \mathfrak{k}} \text{ad}^{\mathfrak{p}}(X_i) \otimes \Theta^i \in \text{End}(\mathfrak{p}) \otimes \wedge^2(\mathfrak{p}^*), \quad (8.2.33)$$

$$\Theta_V = - \sum_{i=1}^{\dim \mathfrak{k}} \pi_*(X_i) \otimes \Theta^i \in \text{End}(V) \otimes \wedge^2(\mathfrak{p}^*). \quad (8.2.34)$$

**8.2.35 LEMMA.** Let  $\Theta^i$  be defined as in Definition 8.2.31. Under the algebra isomorphism  $\wedge(\mathfrak{g}) \rightarrow \wedge(\mathfrak{g}^*)$ ,  $X^* \mapsto X$ , we have

$$\lambda^{\mathfrak{p}}(X_i) \mapsto \Theta^i.$$

*Proof.* Since the 2-form  $\Theta^i$  is in  $\wedge^2(\mathfrak{p}^*)$ , we may write it as

$$\Theta^i = \frac{1}{2} \sum_{a,b=1}^{\dim \mathfrak{p}} \Theta^i(Y_a, Y_b) Y_a^* Y_b^*.$$

Equation 6.1.15 tells us that

$$\Theta^i(Y_a, Y_b) = -X_i^*([Y_a, Y_b]_{\mathfrak{g}}) = -\langle X_i, [Y_a, Y_b]_{\mathfrak{g}} \rangle.$$

Therefore,

$$\Theta^i = -\frac{1}{2} \sum_{a,b=1}^{\dim \mathfrak{p}} \langle X_i, [Y_a, Y_b]_{\mathfrak{g}} \rangle Y_a^* Y_b^*.$$

Comparing this with Equation 8.2.3, we see that the above is exactly the image of  $\lambda^p(X_i)$  under the isomorphism  $\wedge(\mathfrak{g}) \rightarrow \wedge(\mathfrak{g}^*)$ ,  $X \mapsto X^*$ .  $\square$

8.2.36 DEFINITION. We define the algebra homomorphism

$$\mathcal{A}: C^\infty(\mathfrak{k}) \otimes \mathbb{C} \rightarrow \wedge(\mathfrak{p}^*) \otimes \mathbb{C}$$

as the  $\mathbb{C}$ -linear extension of the composition

$$C^\infty(\mathfrak{k}) \rightarrow \wedge(\mathfrak{p}^*) \rightarrow \wedge(\mathfrak{p}),$$

where the first map is given by  $f \mapsto f(-\lambda^p X)$ , and the second map is the algebra isomorphism induced by  $Y^* \mapsto Y$ .

8.2.37 PROPOSITION. Let  $\varphi$  be the element in  $C^\infty(\mathfrak{k}) \otimes \mathbb{C}$  defined by the following power series:

$$\varphi(X) := j_{\mathfrak{k}}(X) j_{\mathfrak{g}/\mathfrak{k}}^{-1}(X) \operatorname{tr}_V(e^{-\pi_* X}).$$

Here  $\operatorname{tr}_V$  denotes the usual trace for the operators on the representation space  $V$  of  $\pi$ . Then,

$$\mathcal{A}(\varphi) = \det^{1/2} \left( \frac{\Theta_T/2}{\sinh \Theta_T/2} \right) \operatorname{tr}_V(e^{-\Theta_V}). \quad (8.2.38)$$

*Proof.* By definition (Equation 8.2.23),

$$j_{\mathfrak{g}/\mathfrak{k}}(X) = \det^{1/2} \left( \frac{\sinh(\operatorname{ad}_X^p/2)}{\operatorname{ad}_X^p/2} \right).$$

Then, by Lemma 8.2.35 and Equation 8.2.33,

$$\mathcal{A}(j_{\mathfrak{g}/\mathfrak{k}}^{-1}) = \det^{1/2} \left( \frac{\Theta_T/2}{\sinh \Theta_T/2} \right). \quad (8.2.39)$$

By the same token,

$$\mathcal{A}(\operatorname{tr}_V(e^{-\pi_*})) = \operatorname{tr}_V(e^{-\Theta_V}).$$

It remains to show that  $\mathcal{A}(j_{\mathfrak{k}}) = 1$ . Because  $j_{\mathfrak{k}}(X)$  is invariant under the  $\operatorname{Ad}(K)$ -action on  $\mathfrak{k}$ , its value is completely determined by  $j_{\mathfrak{k}}(H)$  for  $H$  in a maximal abelian subalgebra of  $\mathfrak{k}$ ; such a subalgebra is the Lie algebra  $\mathfrak{t}$  of a maximal torus of  $K$ . Hence, it is sufficient to show that  $j_{\mathfrak{k}}(-\lambda^p H) = 1$  for  $H$  in  $\mathfrak{t}$ . Let  $\Phi^+$  denote the selected set of positive roots of  $\mathfrak{k}$ . Then

$$j_{\mathfrak{k}}(H) = \prod_{\alpha \in \Phi^+} \frac{\sinh \alpha(H/2)}{\alpha(H/2)}.$$

Let  $\{H_i\}_{i=1}^{\dim \mathfrak{t}}$  be an orthonormal basis for  $\mathfrak{t}$ . Writing  $H = \sum_{i=1}^{\dim \mathfrak{t}} x_i H_i$ , we have

$$j_{\mathfrak{t}}(H) = \prod_{\alpha \in \Phi^+} \frac{\sinh(\sum_{i=1}^{\dim \mathfrak{t}} x_i \alpha(H_i)/2)}{\sum_{i=1}^{\dim \mathfrak{t}} x_i \alpha(H_i)/2}.$$

Thus,

$$\begin{aligned} j_{\mathfrak{t}}(-\lambda^p H) &= \prod_{\alpha \in \Phi^+} \frac{\sinh(\sum_{i=1}^{\dim \mathfrak{t}} \lambda^p(H_i) \alpha(H_i)/2)}{\sum_{i=1}^{\dim \mathfrak{t}} \lambda^p(H_i) \alpha(H_i)/2} \\ &= \prod_{\alpha \in \Phi^+} \frac{\sinh(\lambda^p(\sum_{i=1}^{\dim \mathfrak{t}} \alpha(H_i) H_i)/2)}{\lambda^p(\sum_{i=1}^{\dim \mathfrak{t}} \alpha(H_i) H_i)/2} = \prod_{\alpha \in \Phi^+} \frac{\sinh(\lambda^p(\tilde{H}))}{\lambda^p(\tilde{H})}, \end{aligned}$$

where  $\tilde{H} := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{t}} \alpha(H_i) H_i$ . Since the power series of  $\sinh(x)/x$  involves only the even powers of  $x$  and  $\lambda^p(\tilde{H})\lambda^p(\tilde{H}) = 0$ , we have  $j_{\mathfrak{t}}(-\lambda^p H) = 1$ .  $\square$

**8.2.40 THEOREM (THE LOCAL INDEX THEOREM FOR  $G/K$ ).** *Let  $G$  be a compact connected Lie group equipped with a bi-invariant metric. Let  $K$  be a closed connected Lie subgroup of  $G$  such that  $G/K$  is an even-dimensional spin manifold. Let  $\mathfrak{g}$  and  $\mathfrak{k}$  be the Lie algebra of  $G$  and  $K$ , respectively, and let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ . Consider the Kostant-Dirac operator  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  on the twisted spinor bundle  $G \times_K (\mathbb{S} \otimes V) \rightarrow G/K$ , where  $\mathbb{S}$  is the spinor space for  $\mathbb{C}l(\mathfrak{p})$  and  $V$  is an irreducible representation space for  $K$ . Let  $q_t$  be the heat convolution kernel of  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}^2$ . The leading nonzero term of the asymptotic expansion of the super-trace of  $q_t$ , multiplied by the Riemannian volume form of  $G/K$ , is equal to the top degree part of the product of the Hirzebruch  $\hat{A}$ -class of  $G/K$  and the Chern character of the twisting bundle  $G \times_K V$ ; that is,*

$$\text{Str}(q_t) \text{ vol} = \hat{A}(G/K) \text{ ch}(G \times_K V)|^{\text{top}} + O(t),$$

for  $t \rightarrow 0+$ .

*Proof.* By the homogeneity of  $G/K$ , it is sufficient to check for  $\text{Str}(q_t)$  at  $\bar{e} := eK$ . Let  $\tilde{q}_t$  be the convolution kernel on  $G$  that is equivalent to  $q_t$  (see Corollary 8.2.20); then,

$$\text{Str}(q_t(\bar{e})) = \text{Str}(\tilde{q}_t(e)). \quad (8.2.41)$$

By Lemma 8.2.24 and Equation 8.2.2, we have

$$\tilde{q}_t(e) \sim e^{t\|\rho_{\mathfrak{k}} + \mu\|^2/2} \int_{\mathfrak{t}} h_{\mathfrak{t}}^{\mathfrak{g}}(X) j_{\mathfrak{t}}(X) j_{\mathfrak{g}/\mathfrak{k}}^{-1}(X) e^{-\pi_*(X)} e^{-\gamma^p(X)} dX.$$

By definition,  $\gamma^p = q \circ \lambda^p$ , where  $q$  is the Chevalley map  $\wedge(\mathfrak{p}) \rightarrow \mathbb{C}l(\mathfrak{p})$ . Recall that  $q$  is an algebra homomorphism modulo terms of lower filtration order. As we have seen in Equation 8.1.17, the super-trace is dependent only on the elements with top filtration order. So, as far as the super-trace is concerned, we may replace  $e^{-\gamma^p(X)}$  in the



integrand with  $e^{-\lambda^p(X)}$ . Thus,

$$\begin{aligned} \text{Str}(q_t(\bar{e})) &= \text{Str}(\tilde{q}_t(e)) \\ &\sim \frac{e^{t\|\rho_{\mathfrak{k}}+\mu\|^2/2}}{(2\pi t)^{\dim \mathfrak{p}/2}} \int_{\mathfrak{k}} h_{\mathfrak{k}}^{\mathfrak{k}}(X) j_{\mathfrak{k}}(X) j_{\mathfrak{g}/\mathfrak{k}}^{-1}(X) \text{tr}_V(e^{-\pi_*(X)}) \text{tr}_s(e^{-\lambda^p(X)}) dX, \end{aligned}$$

where  $\text{tr}_s$  is the super-trace over  $\mathbb{S}$ , and  $\text{tr}_V$  is the ordinary trace over  $V$ . Applying Lemma 8.2.25 with

$$\varphi(X) := j_{\mathfrak{k}}(X) j_{\mathfrak{g}/\mathfrak{k}}^{-1}(X) \text{tr}_V(e^{-\pi_* X})$$

leads us to the following conclusions:

(i) We have the asymptotic expansion

$$\text{Str}(q_t(\bar{e})) \sim \frac{e^{t\|\rho_{\mathfrak{k}}+\mu\|^2/2}}{(2\pi t)^{\dim(\mathfrak{p})/2}} \sum_{n=0}^{\infty} t^n \text{tr}_s(\Psi_n), \quad (8.2.42)$$

where  $\Psi_n$  is contained in  $\bigoplus_{\ell=0}^n \wedge^{2\ell}(\mathfrak{p})$ .

(ii) Since the super-trace of an element in  $\wedge^k \mathfrak{p}$  can be nonzero only if  $k$  is the top degree, which means  $k = \dim \mathfrak{p}$  in our case, the leading nonzero term in the asymptotic expansion 8.2.42 comes from the term with  $n = \dim(\mathfrak{p})/2$ . Hence,

$$\text{Str}(q_t(\bar{e})) = \frac{1}{(2\pi)^N} \text{tr}_s(\Psi_N) + O(t), \quad N := \frac{\dim \mathfrak{p}}{2}. \quad (8.2.43)$$

(iii) Owing to Equation 8.2.26,

$$\text{tr}_s(\Psi_N) = \text{tr}_s(\varphi(-\lambda^p X)). \quad (8.2.44)$$

Now the super-trace picks out the top degree part; precisely, by Equation 8.1.17,

$$\text{tr}_s(\Psi_N) Y_1 \cdots Y_{2N} = (-i)^N \Psi_N^{\text{top}}.$$

Hence, Equation 8.2.44 implies

$$\text{tr}_s(\Psi_N) Y_1 \cdots Y_{2N} = \text{tr}_s(\varphi(-\lambda^p X)) Y_1 \cdots Y_{2N} = (-i)^N \varphi(-\lambda^p X)^{\text{top}}.$$

Applying the algebra isomorphism  $\wedge(\mathfrak{p}) \rightarrow \wedge(\mathfrak{p}^*)$ ,  $Y \mapsto Y^*$ , we get

$$\text{tr}_s(\Psi_N) Y_1^* \cdots Y_{2N}^* = (-i)^N \mathcal{A}(\varphi)^{\text{top}}.$$

Since  $Y_1^* \cdots Y_{2N}^*$  is the Riemannian volume form at  $\bar{e}$ , we have

$$\frac{1}{(2\pi)^N} \text{tr}_s(\Psi_N) \text{vol}_{\bar{e}} = \frac{1}{(2\pi i)^N} \mathcal{A}(\varphi)^{\text{top}}.$$

By Proposition 8.2.37,

$$\begin{aligned} \frac{1}{(2\pi)^N} \text{tr}_s(\Psi_N) \text{vol}_{\bar{e}} &= \frac{1}{(2\pi i)^N} \left[ \det^{1/2} \left( \frac{\Theta_T/2}{\sinh \Theta_T/2} \right) \text{tr}_V e^{-\Theta_V} \right]_{\bar{e}}^{\text{top}} \\ &= \hat{A}(G/K) \text{ch}(G \times_K V) \Big|_{\bar{e}}^{\text{top}}. \end{aligned} \quad (8.2.45)$$

The theorem now follows from Equation 8.2.43.  $\square$

# 9

## THE DISTRIBUTIONAL INDEX OF THE CUBIC DIRAC OPERATOR

THE relative Dirac operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ , identified as a differential operator following the prescription in Section 6.3, is a priori, elliptic only in the directions transverse to the  $K$ -orbits in  $G$ . Hence, the kernel of  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ , which is a  $K$ -representation space, may not be finite-dimensional. It turns out, however, that for each irreducible representation  $\mathfrak{u}$  of  $K$ , the multiplicity  $n_{\mathfrak{u}}$  of the representation space  $V_{\mathfrak{u}}$  of  $\mathfrak{u}$  in  $\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})$  is finite [6, Thm. 2.2, p. 10]. So we can associate to  $\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})$  an infinite formal sum  $\sum_{\mathfrak{u} \in \hat{K}} n_{\mathfrak{u}} [V_{\mathfrak{u}}]$  in the formal representation group  $\hat{R}(K) = \prod_{\mathfrak{u} \in \hat{K}} \mathbb{Z} \cdot [V_{\mathfrak{u}}]$ . (We had a brief encounter with  $\hat{R}(K)$  on page 37.) Assuming that  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  is graded, we write

$$[\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})_+] := \sum_{\mathfrak{u} \in \hat{K}} n_{\mathfrak{u}}^+ [V_{\mathfrak{u}}], \quad [\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})_-] := \sum_{\mathfrak{u} \in \hat{K}} n_{\mathfrak{u}}^- [V_{\mathfrak{u}}],$$

where  $n_{\mathfrak{u}}^+$  and  $n_{\mathfrak{u}}^-$  are the multiplicities of  $V_{\mathfrak{u}}$  in the even and odd subspaces of  $\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})$ , respectively. Then, set

$$\text{Ind}_K \mathcal{D}(\mathfrak{g}, \mathfrak{k}) := [\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})_+] - [\ker \mathcal{D}(\mathfrak{g}, \mathfrak{k})_-] = \sum_{\mathfrak{u} \in \hat{K}} (n_{\mathfrak{u}}^+ - n_{\mathfrak{u}}^-) [V_{\mathfrak{u}}].$$

According to the results of M. Atiyah and I. M. Singer [6, Thm. 2.2, p. 10, Thm. 4.6, p. 34], this sum converges, in the distributional sense, to a distribution on  $K$  that is supported at the identity. Hence, we may apply the inverse of the Duflo isomorphism to  $\text{Ind}_K \mathcal{D}(\mathfrak{g}, \mathfrak{k})$  and obtain a distribution  $\text{Duf}^{-1} \text{Ind}_K \mathcal{D}(\mathfrak{g}, \mathfrak{k})$  on  $\mathfrak{k}$  that is supported at 0. Our goal is to prove Theorem 9.2.17, which says that, for any  $\text{ad}(\mathfrak{k})$ -invariant polynomial  $\phi$  on  $\mathfrak{k}$ ,

$$\langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}(\mathfrak{g}, \mathfrak{k}), \phi \rangle = \langle \hat{A} \smile \hat{\phi}, [G/K] \rangle,$$

where  $[G/K]$  is the fundamental homology class of  $G/K$ , and  $\hat{A} \smile \hat{\phi}$  is the cup product of the Hirzebruch  $\hat{A}$ -class of  $G/K$  and the characteristic class  $\hat{\phi}$  of the principal  $K$ -bundle  $G \rightarrow G/K$  given by the image of  $\phi$  under the Chern-Weil homomorphism  $S(\mathfrak{k})^{\mathfrak{k}} \rightarrow H^*(G/K)$ .

## 9.1 THE DISTRIBUTIONAL INDEX OF A TRANSVERSALLY ELLIPTIC OPERATOR

9.1.1 We review, in this section, the basics of transversally elliptic operators. Main reference is [6].

9.1.2 NOTATION. Throughout this chapter,  $K$  is a compact Lie group and  $M$  is a manifold with a  $K$ -action. The cotangent bundle of  $M$  shall be denoted by

$$\xi : T^*M \rightarrow M.$$

And we denote by  $D$  a generic  $K$ -equivariant differential operator on a  $K$ -equivariant<sup>1</sup> vector bundle  $F \rightarrow M$ .

9.1.3 DEFINITION. We define the subset  $T_K^*M$  of  $T^*M$  as

$$T_K^*M := \{ \alpha \in T^*M \mid \alpha(\tilde{X}_{\xi(\alpha)}) = 0, \forall X \in \mathfrak{k} \}.$$

Here  $\tilde{X}$  denotes the fundamental vector field on  $M$  generated by  $X$  (see Equation 6.1.4).

*Remark.* The restriction of  $\xi$  to  $T_K^*M$  makes it a fiber bundle over  $M$ ; each fiber is a vector space, yet the dimension may vary from point to point on  $M$ .

9.1.4 DEFINITION. A  $K$ -equivariant differential operator  $D$  on a  $K$ -equivariant vector bundle  $F \rightarrow M$  is *transversally elliptic* if its principal symbol (which is a bundle morphism  $\xi^*E \rightarrow \xi^*E$ ) is invertible on  $T_K^*M$  outside (the image of) the zero section.

*Remark.* (1) The definition of the principal symbol using pseudodifferential operator theory can be found in [6, Lecture 1].

(2) Elliptic operators are always transversally elliptic. If  $K$  is finite, then transversal ellipticity is equivalent to ellipticity.

9.1.5 By the standard theory of pseudodifferential operators, a transversally elliptic operator  $D : \Gamma(F) \rightarrow \Gamma(F)$  extends to an operator on Sobolev spaces; see [90, Thm. 5.1, p. 47].

9.1.6 THEOREM. *Let  $D$  be a  $K$ -equivariant differential operator on a  $K$ -equivariant vector bundle  $F \rightarrow M$ . If  $D$  is transversally elliptic, then,*

<sup>1</sup> A  $K$ -equivariant vector bundle is a vector bundle  $F \rightarrow M$  such that (i) the total space  $F$  and the base space  $M$  are  $K$ -manifolds, (ii) the projection  $F \rightarrow M$  is  $K$ -equivariant, and (iii) the  $K$ -action on the fibers are linear.

for each  $u$  in  $\widehat{K}$ , the  $u$ -isotypic component of  $\ker D$  and  $\ker D^*$  are finite-dimensional and consist of smooth sections. If we denote the multiplicities of  $u$  in  $\ker D$  and  $\ker D^*$  by  $n_u^+$  and  $n_u^-$ , respectively, and denote the character of  $u$  by  $\chi_u$ , then the sum

$$\sum_{u \in \widehat{K}} (n_u^+ - n_u^-) \chi_u$$

converges as a distribution on  $K$ .

*A Comment on the Proof.* This theorem is from [6, Thm. 2.2, p. 10]; the actual statement there is more general than what we wrote down above. We will not repeat the proof here, but introduce some ideas and constructions that are involved. Let  $\{Y_i\}_{i=1}^{\dim \mathfrak{k}}$  be an orthonormal basis for  $\mathfrak{k}$ . Let  $\check{Y}_i$  denote the fundamental vector field on  $M$  generated by  $Y_i$ . Define the differential operator  $\Delta_K$  on  $M$  as

$$\Delta_K := \sum_{i=1}^{\dim \mathfrak{k}} \check{Y}_i \check{Y}_i.$$

Consider the operator

$$\begin{aligned} A : \Gamma(F) &\rightarrow \Gamma(F) \oplus \Gamma(F) \\ \sigma &\mapsto (D\sigma, (1 - \Delta_K)\sigma). \end{aligned}$$

Because the symbol of  $1 - \Delta_K$  is injective in the direction of  $K$ -orbits, the symbol of the operator  $A$  is injective. Then,

$$A^*A = D^*D + (1 - \Delta_K)^2$$

is an elliptic operator on  $\Gamma(F)$ . Define

$$\begin{aligned} (\ker D)_\lambda &= \{ \sigma \in \Gamma(F) \mid A\sigma = 0 \oplus \lambda\sigma \} \\ &= \{ \sigma \in \Gamma(F) \mid D\sigma = 0, \Delta_K\sigma = (1 - \lambda)\sigma \}. \end{aligned}$$

Then, for any  $\sigma$  in  $(\ker D)_\lambda$ , we have

$$(A^*A)\sigma = \lambda^2\sigma.$$

Thus,  $(\ker D)_\lambda$  is contained in the  $\lambda^2$ -eigenspace of  $A^*A$ . But the eigenspaces of  $A^*A$  are finite-dimensional and consist of smooth sections, owing to the ellipticity of  $A^*A$  [6, Lem. 2.3, p. 10]; hence, so must the subspaces  $(\ker D)_\lambda$  be. Now  $(\ker D)_\lambda$  contains all the  $u$ -isotypic components of  $\ker D$  on which the Casimir element acts by the scalar  $\lambda$  (see Sections 2.2.14 and 2.2.24). Therefore, we conclude that the multiplicities of all irreducible representations of  $K$  in  $\ker D$  must be finite. A similar argument works for  $\ker D^*$ .  $\square$

**9.1.7 DEFINITION.** Owing to Theorem 9.1.6,  $\ker D$  and  $\ker D^*$  define the following elements in the formal representation group  $\widehat{R}(K) = \prod_{u \in \widehat{K}} \mathbb{Z} \cdot [V_u]$ :

$$[\ker D] = \sum_{u \in \widehat{K}} n_u^+ [V_u], \quad [\ker D^*] = \sum_{u \in \widehat{K}} n_u^- [V_u]. \quad (9.1.8)$$

Their difference,

$$\text{Ind}_K D := [\ker D] - [\ker D^*], \quad (9.1.9)$$

is called the *distributional index* of  $D$ .

*Remark.* If  $D$  is elliptic, then  $[\ker D]$  and  $[\ker D^*]$  are finite sums, and they lie in the character ring  $R(K)$ . So their difference  $\text{Ind}_K D$  is also an element of  $R(K)$ . In this case  $\text{Ind}_K D$  is known as the *equivariant index* of  $D$ .

9.1.10 Another way to define the distributional index is to specify its pairing with the test functions  $f$  in  $C^\infty(K)$ . Let  $R : K \rightarrow \text{Aut}(\Gamma(F))$  be the  $K$ -action on the space of smooth sections of  $F$  with respect to which the operator  $D$  is equivariant. Define the operator  $R_f$  that acts on  $\Gamma(F)$  by

$$(R_f \sigma)(x) = \int_K f(k) (R_k \sigma)(x) dk. \quad (9.1.11)$$

Let  $R_f^+$  and  $R_f^-$  be the restrictions of  $R_f$  to  $\ker D$  and  $\ker D^*$ , respectively. Then the pairing of  $\text{Ind}_K D$  with  $f$  in  $C^\infty(K)$  is given by

$$\langle \text{Ind}_K D, f \rangle = \text{tr}(R_f^+) - \text{tr}(R_f^-). \quad (9.1.12)$$

See [6, p. 9–10] for more discussions on this.

*Remark.* Let  $(\cdot, \cdot)$  be a smooth fiber-wise inner product on the vector bundle  $F \rightarrow M$ . Let  $\mu$  be any density on  $M$ . Define an  $L^2$ -inner product on  $\Gamma(F)$  by

$$\langle \sigma_1, \sigma_2 \rangle = \left[ \int_M (\sigma_1(x), \sigma_2(x)) \mu(x) \right]^{1/2}.$$

We may assume that this inner product is  $K$ -invariant by the usual trick of averaging the inner product over  $K$  (see Section 2.2.2). Then,

$$\langle R_k \sigma, R_k \sigma \rangle = \langle \sigma, \sigma \rangle$$

for any  $\sigma$  in  $\Gamma(F)$  and  $k$  in  $K$ ; as a consequence, the linear map  $R_k$  extends, as a unitary operator, to the  $L^2$ -closure  $\Gamma^2(F)$  of  $\Gamma(F)$ . Then, for each  $f$  in  $C^\infty(K)$ , the operator  $R_f$  is a bounded operator; indeed, for any  $\sigma$  in  $\Gamma(F)$  and  $x$  in  $M$ , we have

$$|R_f \sigma(x)| \leq \int_K |f(k)| \|R_k \sigma\| dk,$$

which implies

$$\|R_f\| \leq \int_K |f(k)| dk.$$

The following theorem is from [6, Thm. 4.6, p. 34]:

9.1.13 THEOREM. *The support of the distribution  $\text{Ind}_K D$  is contained in the subset of  $K$  comprising of all elements that admit at least one*

fixed point in  $M$ ; put in another way,

$$\text{supp}(\text{Ind}_K D) = \bigcup_{x \in M} \{h \in K \mid x = x \cdot h\}. \quad (9.1.14)$$

9.1.15 COROLLARY. *If  $K$  acts freely on  $M$  then  $\text{Ind}_K D$  is supported at the identity of  $K$ .*

*Example.* Let  $M$  be a compact connected Lie group  $G$ . Let  $K$  be a maximal torus  $T$  of  $G$ , and let it act on  $G$  on the left. Let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to some  $\text{Ad}(G)$ -invariant inner product. Let  $\mathbb{S}$  be the spinor space for the Clifford algebra  $\text{Cl}(\mathfrak{p})$ . Take the trivial bundle  $G \times \mathbb{S} \rightarrow G$ , and let  $T$  act on the sections by the left-regular action. That means, for  $h$  in  $T$  and  $\sigma$  in  $\Gamma(G \times \mathbb{S})$ ,

$$h \cdot \sigma : g \mapsto \sigma(h^{-1}g).$$

Let  $\mathcal{D}$  be the cubic Dirac operator 5.2.10; following exactly what we have done in Section 6.3.5, we identify  $\mathcal{D}$  as a  $T$ -equivariant differential operator on  $\Gamma(G \times \mathbb{S}) = C^\infty(G) \otimes \mathbb{S}$ . As we have discussed in Section 8.2.4,  $G/T$  is even-dimensional and the spinor space is bi-graded:  $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ . This induces a grading on  $\Gamma(G \times \mathbb{S})$ , and the Dirac operator is an odd operator:  $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}$ . We are interested in the distributional index  $\text{Ind}_T \mathcal{D}_+$  of the (transversally) elliptic operator  $\mathcal{D}_+$ . Since  $\mathcal{D}_- = \mathcal{D}_+^*$  (see Section 8.1.1), we have

$$\text{Ind}_T \mathcal{D}_+ = [\ker \mathcal{D}_+] - [\ker \mathcal{D}_-] \in \widehat{R}(T).$$

Let  $\mathbb{C}_\mu$  denote the irreducible complex  $T$ -representation space with weight  $\mu$ . Recall that the correspondence  $\mathbb{C}_\mu \mapsto \mu$  is an isomorphism from  $\widehat{T}$  to the weight lattice  $\Lambda_T$  (see Section 2.3.4). Decomposing the  $T$ -vector space  $\Gamma(G \times \mathbb{S})$  into isotypic components, we get

$$\begin{aligned} \Gamma(G \times \mathbb{S}^\pm) &\simeq \bigoplus_{\mu \in \Lambda_T} \mathbb{C}_\mu \otimes \text{Hom}(\mathbb{C}_\mu, C^\infty(G) \otimes \mathbb{S}^\pm) \\ &\simeq \bigoplus_{\mu \in \Lambda_T} \mathbb{C}_\mu \otimes (C^\infty(G) \otimes \mathbb{S}^\pm \otimes \mathbb{C}_\mu^*)^T. \end{aligned}$$

Thus,

$$[\ker \mathcal{D}_\pm] = \sum_{\mu \in \Lambda_T} \dim \ker(\mathcal{D}_{\mu\pm}) [\mathbb{C}_\mu]$$

where  $\mathcal{D}_\mu$  is the Kostant-Dirac operator on the space of  $T$ -invariant sections of the trivial bundle  $G \times (\mathbb{S} \otimes \mathbb{C}_\mu^*) \rightarrow G$ . Hence,

$$\text{Ind}_T \mathcal{D}_+ = \sum_{\mu \in \Lambda_T} \text{ind}_s \mathcal{D}_\mu [\mathbb{C}_\mu], \quad (9.1.16)$$

where  $\text{ind}_s \mathcal{D}_\mu$  is the usual graded index of  $\mathcal{D}_\mu$ .

Let  $W$  be the Weyl group of  $G$ , and let  $\rho$  be half the sum of the

positive roots of  $G$ . For  $\mu$  in  $\Lambda_T$ , set

$$\text{Alt}(\mu) := \sum_{w \in W} \text{sgn}(w) w \cdot \mu \in \Lambda_T.$$

By the properties of the weight lattice summarized in Sections 2.3.11 and 2.3.15,  $\text{Alt}(\mu)$  is nonzero only if  $W$  acts freely on  $\mu$ , that is,  $\mu$  is off the walls of the Weyl chambers in  $\mathfrak{t}^*$ ; and if that is the case, there is a unique dominant weight  $\lambda$  in  $\Lambda_T$  such that  $\text{Alt}(\lambda + \rho)$  is equal to  $\text{Alt}(\mu)$  up to a sign. According to a result of R. Bott [16, § 6–7],

$$\text{ind}_s \mathcal{P}_\mu = \begin{cases} \dim(V_\lambda), & \text{if } \text{Alt}(\mu) = -\text{Alt}(\lambda + \rho), \\ -\dim(V_\lambda), & \text{if } \text{Alt}(\mu) = \text{Alt}(\lambda + \rho), \\ 0, & \text{if } \text{Alt}(\mu) = 0, \end{cases} \quad (9.1.17)$$

where  $V_\lambda$  is the irreducible representation space of  $G$  with highest weight  $\lambda$ .

As a simple example, let  $G = \text{SU}(2)$ . Then  $T$  is isomorphic to  $U(1)$ ; and we have  $\Lambda_T \simeq \mathbb{Z}$ . In fact, we did some explicit calculations in Section 3.2 and saw that

$$\Lambda_T = \mathbb{Z} \cdot \rho.$$

Let us choose the nonnegative integral multiples of  $\rho$  as the dominant weights. Let  $V_n$  denote the irreducible representation of  $G$  with highest weight  $n\rho$ ,  $n \geq 0$ . Equation 9.1.17, in this case, simplifies to

$$\text{ind}_s \mathcal{P}_{n\rho} = \begin{cases} -\dim(V_{(n-1)\rho}), & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ \dim(V_{-(n+1)\rho}), & \text{if } n < 0. \end{cases}$$

By Equation 3.2.12,

$$\dim(V_{n\rho}) = n + 1.$$

Thus,

$$\text{Ind}_T \mathcal{P}_+ = \sum_{n \in \mathbb{Z}} (-n) [\mathbb{C}_{n\rho}].$$

Let  $H$  be the vector in  $\mathfrak{t}$  such that  $\rho(H) = 1$ . Then, the character of the representation of  $T$  on  $\mathbb{C}_{n\rho}$  is given by

$$\chi_n(\exp(xH)) = e^{inx}.$$

In terms of these characters,

$$\text{Ind}_T \mathcal{P}_+ = - \sum_{n \in \mathbb{Z}} n \chi_n.$$

This is just the derivative of the distribution  $i \sum_{n \in \mathbb{Z}} \chi_n$ ; but  $\sum_{n \in \mathbb{Z}} \chi_n$  is, by Fourier theory, the Dirac delta distribution  $\delta_e$  supported at the identity. Hence,

$$\text{Ind}_T \mathcal{P}_+ = i \delta'_e.$$

## 9.2 THE DISTRIBUTIONAL INDEX OF $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$

9.2.1 We are interested in the case where  $K$  is a closed connected subgroup of a compact connected Lie group  $G$  and the  $K$ -manifold is  $M := G$ , on which  $K$  is acting by right-multiplication. We assume that  $K$  is of maximal rank and that  $G/K$  is a spin manifold. We choose a bi-invariant metric on  $G$  such that the volume form it generates for  $G$  and  $K$  assign unit volume to them.

We continue to use the notations set up in Section 7.2.1. But for simplicity, we assume that the  $\mathrm{Cl}(\mathfrak{p})$ -module  $E$  is just the spinor space  $\mathbb{S}$ . Then  $K$  acts on  $\mathbb{S}$  by  $\nu := \widetilde{\mathrm{Ad}}$ , which is the lift of  $K \xrightarrow{\mathrm{Ad}} \mathrm{SO}(\mathfrak{p})$  to  $\mathrm{Spin}(\mathfrak{p})$ .

Consider the trivial bundle  $G \times \mathbb{S} \rightarrow G$ . Denote by  $\Gamma^2(G \times \mathbb{S})$  the  $L^2$ -closure of  $\Gamma(G \times \mathbb{S})$ . We let  $K$  act on  $\Gamma^2(G \times \mathbb{S})$  as follows: For  $k$  in  $K$  and  $\sigma$  in  $\Gamma^2(G \times \mathbb{S})$ ,

$$k \cdot \sigma : g \mapsto \nu(k)\sigma(gk).$$

Following the notation in Section 9.1.10, we denote this representation by

$$R : K \rightarrow \mathrm{Aut}(\Gamma^2(G \times \mathbb{S})).$$

For  $f$  in  $C^\infty(K)$ , the operator  $R_f$  defined by Equation 9.1.11 acts on  $\Gamma^2(G \times \mathbb{S})$  by

$$R_f \sigma : g \mapsto \int_K f(k) \nu(k) \sigma(gk) dk. \quad (9.2.2)$$

The differential operator we have in mind is the relative Dirac operator  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  on the space  $\Gamma(G \times \mathbb{S})^K$  of  $K$ -invariant sections of  $G \times \mathbb{S}$ ; this operator is, a priori, only transversally elliptic (Lemma 9.2.4). But, as we have seen in Proposition 6.3.14,  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  is equivalent to the Dirac operator  $\mathcal{D}_{\mathfrak{g}/\mathfrak{k}}$  on  $\Gamma(G \times_\nu \mathbb{S})$ , so  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  is effectively elliptic. Moreover,  $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$  admits a unique self-adjoint extension  $\Gamma^2(G \times \mathbb{S})^K$  (see Section 8.1.1); we shall denote this extension by  $\mathcal{D}$ .

Because  $\mathfrak{p}$  is even-dimensional, the spinor space is  $\mathbb{Z}/2\mathbb{Z}$ -graded (see Section 5.1.44):

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-.$$

This induces a grading on the space of sections:

$$\Gamma^2(G \times \mathbb{S})^K = \Gamma^2(G \times \mathbb{S}^+)^K \oplus \Gamma^2(G \times \mathbb{S}^-)^K.$$

The Dirac operator  $\mathcal{D}$  is an odd operator (see Section 6.2.6). We write

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

The restriction  $\mathcal{D}_+$  of  $\mathcal{D}$  to the even subspace is also transversally elliptic. The distributional index of  $\mathcal{D}_+$  is

$$\mathrm{Ind}_K \mathcal{D}_+ = [\ker \mathcal{D}_+] - [\ker \mathcal{D}_-].$$



9.2.3 DEFINITION. We define the distributional index of  $\mathcal{D}$  as

$$\text{Ind}_K \mathcal{D} := \text{Ind}_K \mathcal{D}_+ = [\ker \mathcal{D}_+] - [\ker \mathcal{D}_-].$$

9.2.4 LEMMA. *The Dirac operator  $\mathcal{D}$  is transversally elliptic.*

*Proof.* Since  $\mathcal{D}$  is invariant under left-translations on  $G$ , it is sufficient to check its transversal ellipticity at the identity of  $G$ . With respect to the orthogonal decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , the cotangent space of  $G$  at the identity decomposes into  $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$ . Then the fiber of  $T_K^* G$  over the identity can be naturally identified with  $\mathfrak{p}^*$ . Let  $\{Y_i\}_{i=1}^{\dim \mathfrak{p}}$  be an orthonormal basis for  $\mathfrak{p}$ . Then, as we have seen in the proof of Proposition 6.3.14,  $\mathcal{D}$  at the identity can be expressed as

$$\mathcal{D} = \sum_{i=1}^{\dim \mathfrak{p}} c(Y_i) \partial_{Y_i} + \frac{1}{3} \sum_{i=1}^{\dim \mathfrak{p}} c(Y_i) \gamma^{\mathfrak{p}}(Y_i),$$

where  $c$  is the Clifford action and  $\partial_{Y_i}$  denotes the directional derivative with respect to  $Y_i$ . From this expression it is clear that  $\mathcal{D}$  is transversally elliptic.  $\square$

9.2.5 NOTATION. Let  $\Delta_G$  be the Laplacian on  $G$ . We set

$$L := \frac{1}{2} \Delta_G + \frac{1}{2} (\|\rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2).$$

By Equation 7.2.5,  $L$  is equal to  $\mathcal{D}^2$  on the domain of  $\mathcal{D}$ .

9.2.6 LEMMA. *Let  $f$  be a smooth function on  $K$ . Let  $R_f$  be the operator on  $\Gamma^2(G \times \mathbb{S})^K$  defined by Equation 9.2.2. Then*

$$\langle \text{Ind}_K \mathcal{D}, f \rangle = \text{Str}(R_f e^{\text{TL}}),$$

where the super-trace on the right-hand side is over  $\Gamma^2(G \times \mathbb{S})^K$ .

*Proof.* By Equation 9.1.12, we have

$$\langle \text{Ind}_K \mathcal{D}, f \rangle = \text{tr}(R_f^+) - \text{tr}(R_f^-), \quad (9.2.7)$$

We wish to show that

$$\text{tr}(R_f^+) - \text{tr}(R_f^-) = \text{Str}(R_f e^{\text{TL}}). \quad (9.2.8)$$

First, note that  $R_f$  acts as a scalar  $C := \int_K f(k) dk$  on  $\Gamma^2(G \times K)^K$ . So  $\text{tr}(R_f^+) - \text{tr}(R_f^-)$  is equal to  $C$  times the (graded) index of  $L$ . Hence,

$$\text{tr}(R_f^+) - \text{tr}(R_f^-) = C \text{ind}_s L = C \text{Str}(e^{\text{TL}}) = \text{Str}(R_f e^{\text{TL}}). \quad \square$$

9.2.9 PROPOSITION. *Let  $f$  be a smooth function on  $K$ . Let  $r_t$  be the heat convolution kernel of  $L$ . Then*

$$\langle \text{Ind}_K \mathcal{D}, f \rangle = \int_{G/K} \int_K \text{Str}(r_t(k) f(k) \nu(k)^{-1}) dk d\bar{g}$$

where  $\bar{g} := gK$ .

*Proof.* Recall that  $R_f$  acts on  $\Gamma^2(G \times \mathbb{S})^K$  as the scalar  $C := \int_K f(k) dk$ . So

$$\text{Str}(R_f e^{tL}) = C \text{Str}(e^{tL}) = C \int_G \text{tr}_s(R_t(g, g)) dg,$$

where  $R_t$  is the heat kernel of  $L$ . By Equation 7.1.4,  $\text{tr}_s(R_t(g, g)) = \text{tr}_s(R_t(e, e))$ . Since  $r_t(x) = R_t(e, x)$  by definition,  $\text{tr}_s(R_t(g, g)) = \text{tr}_s(r_t(e))$ . Hence,

$$\text{Str}(R_f e^{tL}) = C \int_G \text{tr}_s(r_t(e)) dg = C \int_{G/K} \int_K \text{tr}_s(r_t(e)) dk d\bar{g}.$$

Because the measure on  $K$  is normalized,

$$\text{Str}(R_f e^{tL}) = C \int_{G/K} \text{tr}_s(r_t(e)) d\bar{g}.$$

Since  $C = \int_K f(k) dk$ ,

$$\text{Str}(R_f e^{tL}) = \int_{G/K} \int_K f(k) \text{tr}_s(r_t(e)) dk d\bar{g}.$$

Owing to Lemma 8.2.15,  $r_t(e) = r_t(k)v(k)^{-1}$ ; hence,

$$\text{Str}(R_f e^{tL}) = \int_{G/K} \int_K f(k) \text{tr}_s(r_t(k)v(k)^{-1}) dk d\bar{g}. \quad \square$$

9.2.10 Since  $K$  acts freely on  $G$ , the distributional index  $\text{Ind}_K \mathcal{D}$  is supported at the identity of  $K$  (Corollary 9.1.15). Recall the distribution theoretic description of the Duflo isomorphism for the Lie group  $K$  (see Section 4.1.22):

$$\text{Duf} = \exp_* \circ j_{\mathfrak{k}} : \mathcal{E}'_0(\mathfrak{k}) \xrightarrow{\sim} \mathcal{E}'_e(K). \quad (9.2.11)$$

Here  $\exp_*$  is the pushforward map of distributions along the exponential map,  $j_{\mathfrak{k}}$  is the multiplication by the function defined by the power series 4.1.24,  $\mathcal{E}'_0(\mathfrak{k})$  is the algebra of distributions on  $\mathfrak{k}$  with support  $\{0\}$ , and  $\mathcal{E}'_e(K)$  is the algebra of distributions on  $K$  with support  $\{e\}$ . Since  $\text{Ind}_K \mathcal{D}$  is in  $\mathcal{E}'_e(K)$ , we can apply the inverse of the Duflo isomorphism to it and obtain  $\text{Duf}^{-1} \text{Ind}_K \mathcal{D}$  in  $\mathcal{E}'_0(\mathfrak{k})$ .

Let us examine the pairing of  $\text{Duf}^{-1} \text{Ind}_K \mathcal{D}$  with a test function. To that end, let  $V$  be a neighborhood of 0 in  $\mathfrak{k}$  that is mapped diffeomorphically by the exponential map onto a neighborhood  $U$  of  $e$  in  $K$ ; denote this local diffeomorphism by

$$\exp_V : V \xrightarrow{\sim} U.$$

Let  $\log := \exp_V^{-1}$ . We have the induced pushforward map of distributions,

$$\log_* : \mathcal{D}'(U) \rightarrow \mathcal{D}'(V).$$

This makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{D}'(V) & \xleftarrow[\sim]{\log_*} & \mathcal{D}'(U) \\ \uparrow & & \uparrow \\ \mathcal{E}'_0(\mathfrak{k}) & \xrightarrow[\sim]{\exp_*} & \mathcal{E}'_e(K) \end{array}$$

The horizontal maps are linear isomorphisms. Thus, for  $\delta$  in  $\mathcal{E}'_e(K)$ , we have

$$(\exp_* \circ \log_*)(\delta) = \delta.$$

So

$$(\text{Duf} \circ j_{\mathfrak{k}}^{-1} \circ \log_*)(\delta) = (\exp_* \circ j_{\mathfrak{k}} \circ j_{\mathfrak{k}}^{-1} \circ \log_*)(\delta) = \delta. \quad (9.2.12)$$

Hence,

$$(j_{\mathfrak{k}}^{-1} \circ \log_*)(\delta) = \text{Duf}^{-1}(\delta). \quad (9.2.13)$$

Now let  $\phi$  be a smooth function on  $\mathfrak{k}$ . Let  $\psi$  be any bump function supported on  $V$  such that  $\psi(X) = 1$  for  $X$  near 0 in  $\mathfrak{k}$ . Then, since  $\text{Duf}^{-1} \text{Ind}_K \mathcal{D}$  is supported at 0,

$$\langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}, \phi \rangle = \langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}, \phi \psi \rangle.$$

Then, by Equation 9.2.13,

$$\langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}, \phi \rangle = \langle j_{\mathfrak{k}}^{-1} \log_* \text{Ind}_K \mathcal{D}, \phi \psi \rangle = \langle \text{Ind}_K \mathcal{D}, \log^*(j_{\mathfrak{k}}^{-1} \phi \psi) \rangle.$$

Finally, by Proposition 9.2.9,

$$\langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}, \phi \rangle = \int_{G/K} \int_K \text{tr}_s(\tau_t(k) \log^*(j_{\mathfrak{k}}^{-1} \phi \psi)(k) \nu(k)^{-1}) dk d\bar{g}. \quad (9.2.14)$$

If  $\phi$  is a polynomial on  $\mathfrak{k}$  that is invariant under  $\text{ad}(\mathfrak{k})$ -action, then it is subject to the Chern-Weil homomorphism 6.1.27. Consider the *integral* Chern-Weil homomorphism (this is not a standard terminology; see the remark on page 99 for the reason we call it this way)

$$\begin{aligned} S(\mathfrak{k}^*) &\rightarrow H^*(G/K), \\ \phi &\mapsto \hat{\phi}, \end{aligned} \quad (9.2.15)$$

obtained by putting an extra factor of  $(2\pi i)^{-1}$  in front of the curvature 2-form  $\Theta$  in Equation 6.1.28. Thus, if  $\phi$  is a homogeneous polynomial of degree  $n$ , the differential form  $\hat{\phi}$  at the identity coset  $\bar{e}$  satisfies

$$\hat{\phi}_{\bar{e}} = (2\pi i)^{-n} \mathcal{A}(\phi), \quad (9.2.16)$$

where  $\mathcal{A}$  is the algebra homomorphism defined in Definition 8.2.36. (Here we are identifying the cotangent space of  $G/K$  at  $\bar{e}$  with  $\mathfrak{p}^*$ , where the  $\mathfrak{p}$  is the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ .) We then have the following result (This theorem was suggested by N. Higson):

**9.2.17 THEOREM.** *Let  $\phi$  be an invariant polynomial on  $\mathfrak{k}$ . Let  $\hat{\phi}$  be the characteristic class of the principal  $K$ -bundle  $G \rightarrow G/K$  obtained by the image of  $\phi$  under the integral Chern-Weil homomorphism 9.2.15. Let*

$\hat{A}$  be the Hirzebruch  $\hat{A}$ -class of the tangent bundle of  $G/K$ . Let  $\text{Ind}_K \mathcal{D}$  be the distributional index of the relative Dirac operator on  $\Gamma(G \times \mathbb{S})^K$ . The pairing of  $\phi$  with the inverse image of  $\text{Ind}_K \mathcal{D}$  under the Duflo isomorphism 9.2.11 satisfies

$$\langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}, \phi \rangle = \langle \hat{A} \smile \hat{\phi}, [G/K] \rangle. \quad (9.2.18)$$

Here  $\hat{A} \smile \hat{\phi}$  is the cup product of  $\hat{A}$  and  $\hat{\phi}$ , and  $[G/K]$  is the fundamental homology class of  $G/K$  associated to the quotient measure on  $G/K$  with respect to the normalized Haar measures on  $G$  and  $K$ .

*Proof.* By Equation 9.2.14,

$$\langle \text{Duf}^{-1} \text{Ind}_K \mathcal{D}, \phi \rangle = \int_{G/K} \int_K \text{tr}_s(r_t(k) \log^*(j_{\mathfrak{k}}^{-1} \phi \psi)(k) \nu(k)^{-1}) dk d\bar{g}. \quad (9.2.19)$$

Let us write the integral over  $K$  as

$$I(t) := \int_K \text{tr}_s(r_t(k) \log^*(j_{\mathfrak{k}}^{-1} \phi \psi)(k) \nu(k)^{-1}) dk.$$

We claim that, for  $t \rightarrow 0+$ ,

$$I(t) \text{ vol} = \hat{A} \hat{\phi}|^{\text{top}} + O(t), \quad (9.2.20)$$

where  $\text{vol}$  is the volume form of  $G/K$  associated with the quotient measure on  $G/K$ , and  $\hat{A} \hat{\phi}|^{\text{top}}$  is the top degree part of the exterior product of  $\hat{A}$  and  $\hat{\phi}$ . Assume, for the moment, that the claim is true; then the theorem follows, since the left-hand side of Equation 9.2.19 is independent of  $t$ .

In order to prove our claim, let us change the domain of the integral  $I(t)$ , from  $K$  to  $\mathfrak{k}$ . We have

$$I(t) = \int_{\mathfrak{k}} \text{tr}_s(r_t(\exp X) j_{\mathfrak{k}}(X) \phi(X) \psi(X) e^{-\gamma^p(X)}) dX. \quad (9.2.21)$$

For  $t \rightarrow 0+$ , the convolution kernel  $r_t$  is of  $O(t^\infty)$  outside any neighborhood of the identity; and since  $\psi(X) = 1$  for  $X$  near 0, we have

$$I(t) \sim \int_{\mathfrak{k}} \text{tr}_s(r_t(\exp X) j_{\mathfrak{k}}(X) \phi(X) e^{-\gamma^p(X)}) dX. \quad (9.2.22)$$

By Theorem 7.2.15 and Equation 8.2.22, we have

$$I(t) \sim e^{t\|\rho_{\mathfrak{k}}\|^2/2} \int_{\mathfrak{k}} h_t^g(X) j_{g/k}(X)^{-1} \phi(X) \text{tr}_s(e^{-\gamma^p(X)}) dX.$$

We may replace  $\gamma^p$  with  $\lambda^p$  because the super-trace only cares about the top degree part;

$$I(t) \sim e^{t\|\rho_{\mathfrak{k}}\|^2/2} \int_{\mathfrak{k}} h_t^g(X) j_{g/k}(X)^{-1} \phi(X) \text{tr}_s(e^{-\lambda^p(X)}) dX.$$

Apply Lemmas 8.2.25 and 8.2.35 exactly as done in the proof of Theorem 8.2.40; then, we get

$$I(t) \text{ vol}_{\mathfrak{k}} = (2\pi i)^{-\dim(G/K)} \mathcal{A}(j_{g/\mathfrak{k}}^{-1}) \mathcal{A}(\phi)|_{\mathfrak{k}}^{\text{top}} + O(t).$$

By Equations 8.2.39 and 9.2.16, we have

$$I(t) \operatorname{vol}_{\bar{e}} = \hat{A} \hat{\phi}|_{\bar{e}}^{\operatorname{top}} + O(t). \quad (9.2.23)$$

This is Equation 9.2.20 at  $\bar{e}$ . By homogeneity, it must hold everywhere on  $G/K$ . The theorem now follows.  $\square$

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## VITA

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Seunghun Hong's curiosity in science was aroused by the visit of Halley's comet in 1986; it eventually led him to enter Seoul National University as a physics major. While yet a university student, he fulfilled his obligatory military service at the 121st General Hospital as a radiographic technician. He received his Bachelor of Science in 2002, and entered Tufts University for graduate studies in physics. While taking some graduate courses in mathematics, in particular, a course on abstract algebra taught by Loring W. Tu, he got captivated by the beauty of mathematics and decided to switch his discipline. After receiving his Master of Science in physics in 2004, he began his formal graduate studies in mathematics. He wrote a thesis under the guidance of Loring W. Tu and received a Master of Arts in mathematics in 2006. Intrigued by the work of Alain Connes, he decided to pursue his doctoral studies in the area of noncommutative geometry. He entered the Graduate School at PENN STATE, and became a student of Nigel Higson.

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